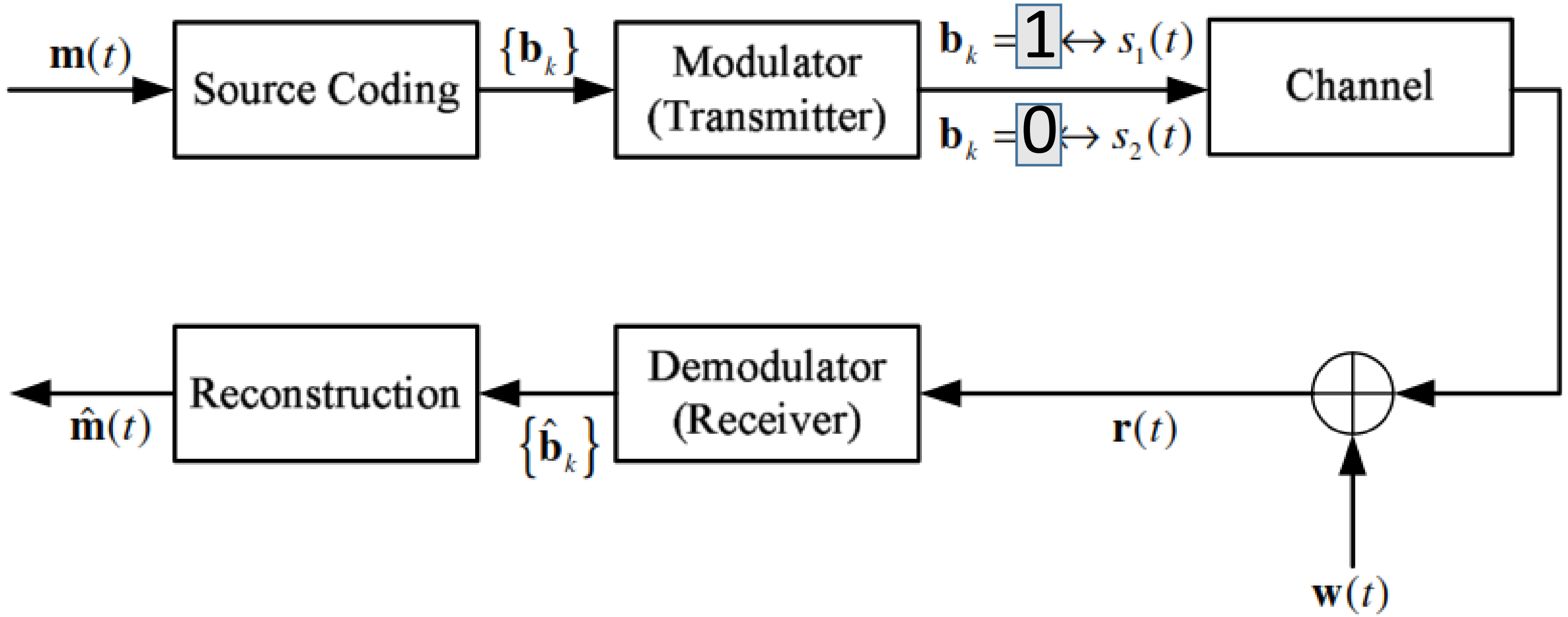
M-Ary Data Transmission

- Orthogonal Functions and Signal Space Representation
- Gram-Schmidt Orthogonaliztion Procedure
- Optimum Receiver for Binary Transmission (Revisited using signal space concept)
- Optimum Receiver for M-Ary Transmission (using signal space representation)
- M-ary Coherent Amplitude-Shift Keying (M-ASK)
- M-ary Coherent Phase-Shift Keying (M-PSK)
- M-ary Coherent Frequency-Shift Keying (M-FSK)
- M-ary Quadrature Amplitude Modulation (M-QAM)
- Union Bound on the Symbol Probability of Error
- Comparison of the various M-ary modulation techniques

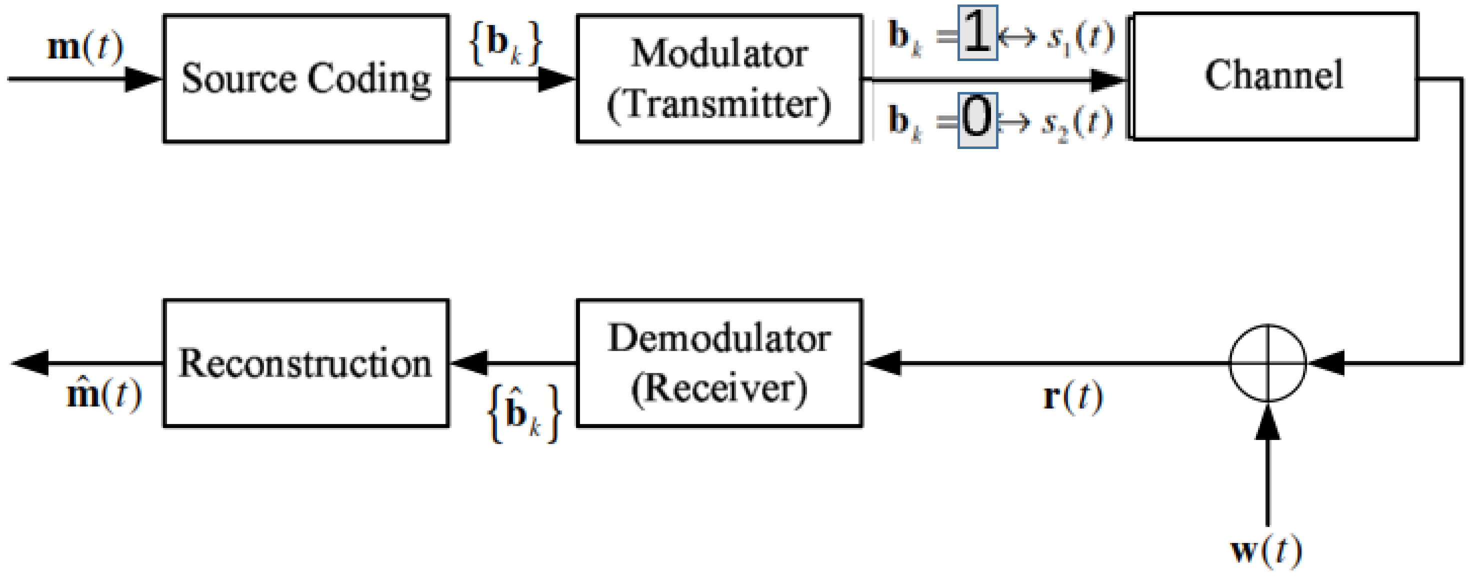
The Binary Communication System (Revisited)



- Bits in two different time slots are *statistically independent*.
- *a priori* probabilities: $P[\mathbf{b}_k = 1] = P_1$, $P[\mathbf{b}_k = 0] = P_2$.
- Signals $s_1(t)$ and $s_2(t)$ have a duration of T_b seconds and finite energies: $E_1 = \int_0^{T_b} s_1^2(t)\mathrm{d}t$, $E_2 = \int_0^{T_b} s_2^2(t)\mathrm{d}t$.
- Noise $\mathbf{w}(t)$ is stationary *Gaussian*, zero-mean *white* noise with two-sided power spectral density of $N_0/2$ (watts/Hz):

$$E\{\mathbf{w}(t)\} = 0, \quad E\{\mathbf{w}(t)\mathbf{w}(t+\tau)\} = \frac{N_0}{2}\delta(\tau).$$

The Binary Communication System (Revisited)



- Received signal over $[(k-1)T_b, kT_b]$:

$$\mathbf{r}(t) = s_i(t-(k-1)T_b) + \mathbf{w}(t), \quad (k-1)T_b \le t \le kT_b.$$

- Objective is to design a receiver (or demodulator) such that *the probability of making an error is minimized.*
- Shall reduce the problem from the observation of a time waveform to that of observing a set of numbers (which are random variables).

Geometric Representation of Signals (Signal Space Concept)

Wish to represent two arbitrary signals $s_1(t)$ and $s_2(t)$ as *linear combinations* of two *orthonormal* basis functions $\phi_1(t)$ and $\phi_2(t)$.

- $\phi_1(t)$ and $\phi_2(t)$ are orthonormal if:

$$\int_0^{T_b} \phi_1(t)\phi_2(t)\mathrm{d}t = 0 \text{ (\textit{orthogonality})},$$

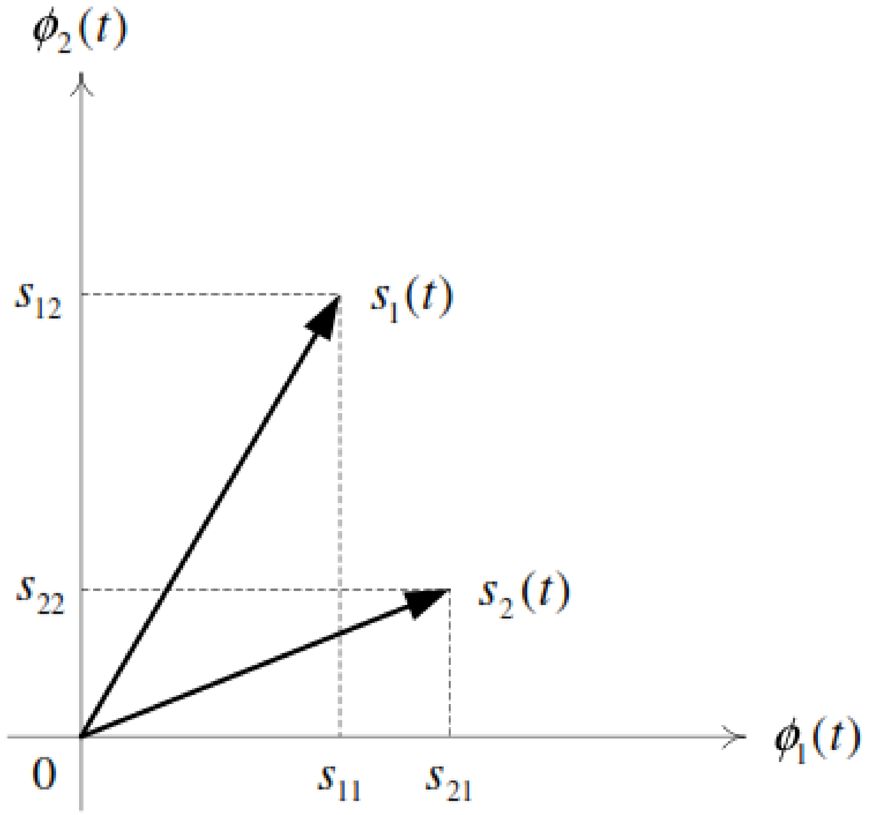
$$\int_0^{T_b} \phi_1^2(t)\mathrm{d}t = \int_0^{T_b} \phi_2^2(t)\mathrm{d}t = 1 \text{ (\textit{normalized} to have unit energy)}.$$

- The representations are

$$\begin{aligned} s_1(t) &= s_{11}\phi_1(t) + s_{12}\phi_2(t), \\ s_2(t) &= s_{21}\phi_1(t) + s_{22}\phi_2(t). \end{aligned}$$

where $\quad s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t, \quad i,j \in \{1,2\},$

Geometric Representation of Signals



$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t),$$
$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t),$$
$$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t,\ i,j \in \{1,2\},$$

$$\boldsymbol{d_{12}^2 = \int_0^{T_b} (s_1(t) - s_2(t))^2 dt}$$
$$\boldsymbol{= (s_{11} - s_{21})^2 + (s_{12} - s_{22})^2}$$

- $\int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t$ is the projection of $s_i(t)$ onto $\phi_j(t)$.
- How to choose *orthonormal* functions $\phi_1(t)$ and $\phi_2(t)$ to represent $s_1(t)$ and $s_2(t)$ exactly?

Gram-Schmidt Method: Binary Case

1. Let $\phi_1(t) \equiv \frac{s_1(t)}{\sqrt{E_1}}$. Note that $s_{11} = \sqrt{E_1}$ and $s_{12} = 0$.
2. Project $s_2'(t) = \frac{s_2(t)}{\sqrt{E_2}}$ onto $\phi_1(t)$ to obtain the *correlation coefficient*:

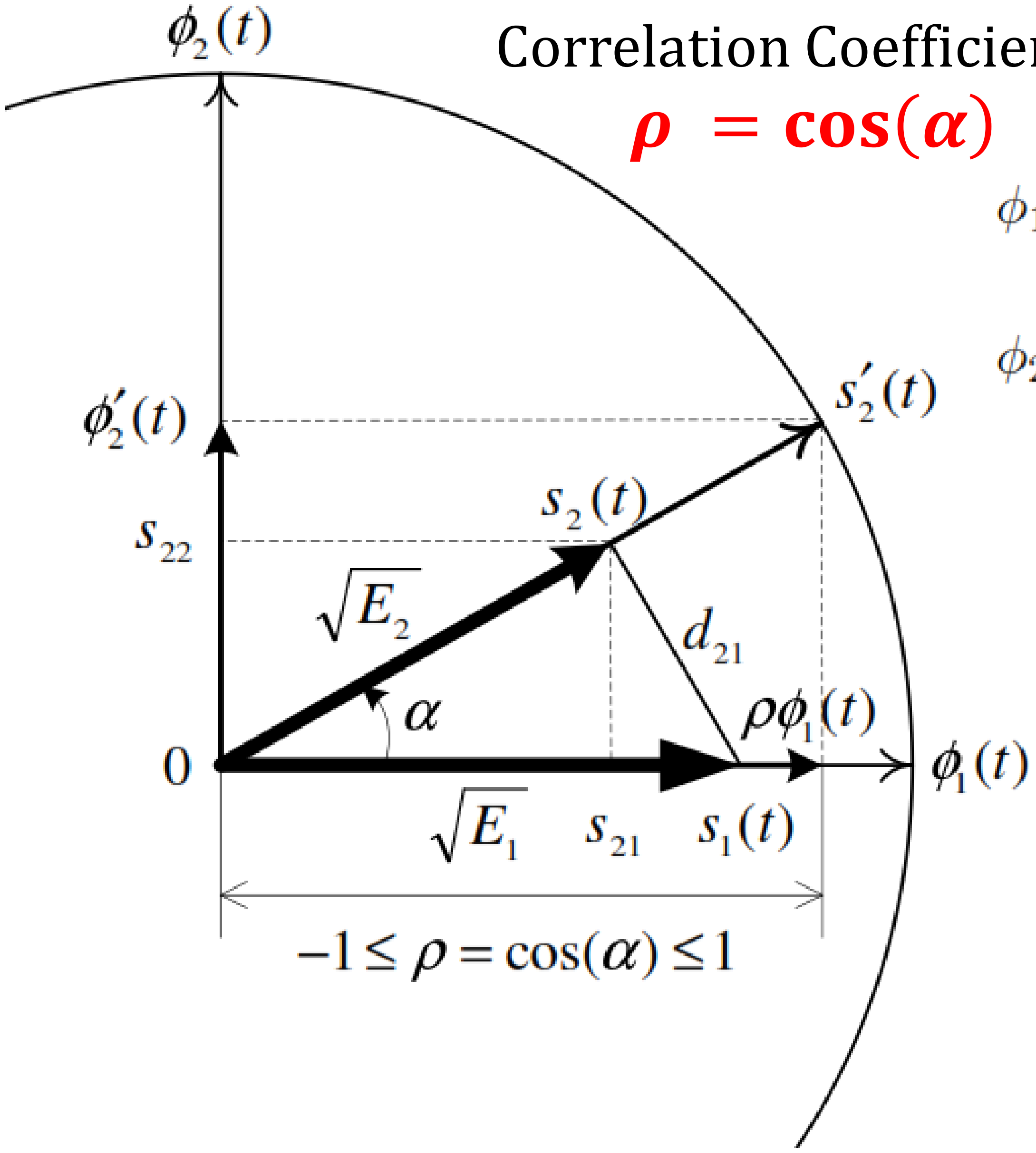
$$\rho = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}} \phi_1(t) \mathrm{d}t = \frac{1}{\sqrt{E_1 E_2}} \int_0^{T_b} s_1(t) s_2(t) \mathrm{d}t.$$

3. Subtract $\rho\phi_1(t)$ from $s_2'(t)$ to obtain $\phi_2'(t) = \frac{s_2(t)}{\sqrt{E_2}} - \rho\phi_1(t)$.
4. Finally, normalize $\phi_2'(t)$ to obtain:

$$\begin{aligned} \phi_2(t) &= \frac{\phi_2'(t)}{\sqrt{\int_0^{T_b} \left[\phi_2'(t)\right]^2 \mathrm{d}t}} = \frac{\phi_2'(t)}{\sqrt{1-\rho^2}} \\ &= \frac{1}{\sqrt{1-\rho^2}} \left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right]. \end{aligned}$$

The Gram-Schmidt method is a procedure for generating a set of **orthonormal functions** from a set of given functions. The original set of functions may be dependent or independent, but the orthogonal functions are linearly independent and orthonormal.

Gram-Schmidt Method: Binary Case



Correlation Coefficient

$$\rho = \cos(\alpha)$$

$$\begin{aligned}
\phi_1(t) &= \frac{s_1(t)}{\sqrt{E_1}}, \\
\phi_2(t) &= \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right], \\
s_{21} &= \int_0^{T_b} s_2(t)\phi_1(t)\mathrm{d}t = \rho\sqrt{E_2}, \\
s_{22} &= \left(\sqrt{1-\rho^2}\right)\sqrt{E_2}, \\
d_{21} &= \sqrt{\int_0^{T_b}[s_2(t)-s_1(t)]^2\mathrm{d}t} \\
&= E_1 - 2\rho\sqrt{E_1E_2} + E_2.
\end{aligned}$$

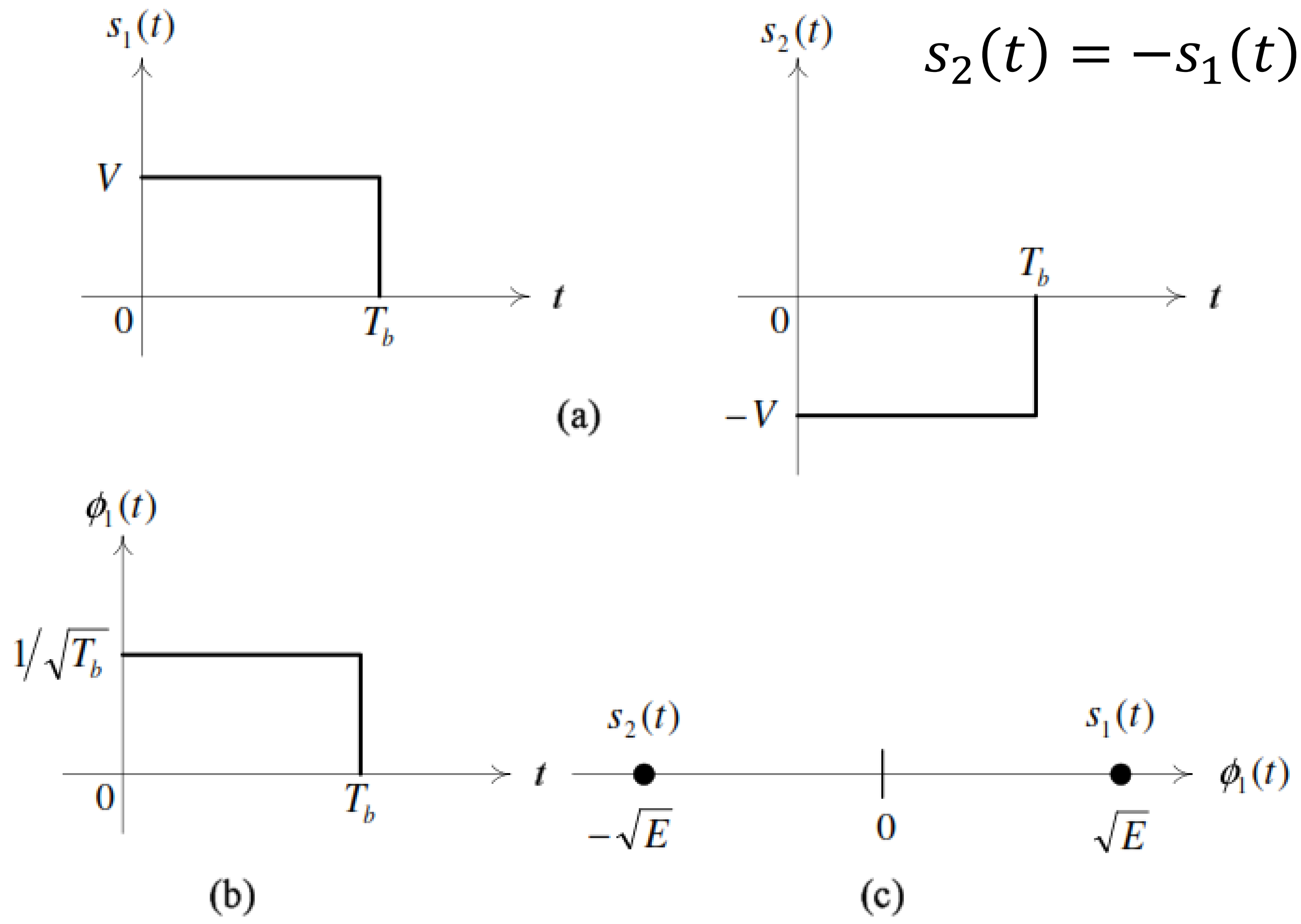
Gram-Schmidt Method: M-Ary Case

$$\begin{aligned}
\phi_1(t) &= \frac{s_1(t)}{\sqrt{\int_{-\infty}^{\infty} s_1^2(t)\mathrm{d}t}}, \\
\phi_i(t) &= \frac{\phi_i'(t)}{\sqrt{\int_{-\infty}^{\infty} \left[\phi_i'(t)\right]^2 \mathrm{d}t}}, \quad i = 2, 3, \ldots, N, \\
\phi_i'(t) &= \frac{s_i(t)}{\sqrt{E_i}} - \sum_{j=1}^{i-1} \rho_{ij}\phi_j(t), \\
\rho_{ij} &= \int_{-\infty}^{\infty} \frac{s_i(t)}{\sqrt{E_i}}\phi_j(t)\mathrm{d}t, \quad j = 1, 2, \ldots, i-1.
\end{aligned}$$

If the waveforms $\{s_i(t)\}_{i=1}^{M}$ form a *linearly independent set*, then $N = M$. Otherwise $N < M$.

Example: Polar Non-return to zero Binary Signals

$$s_2(t) = -s_1(t)$$



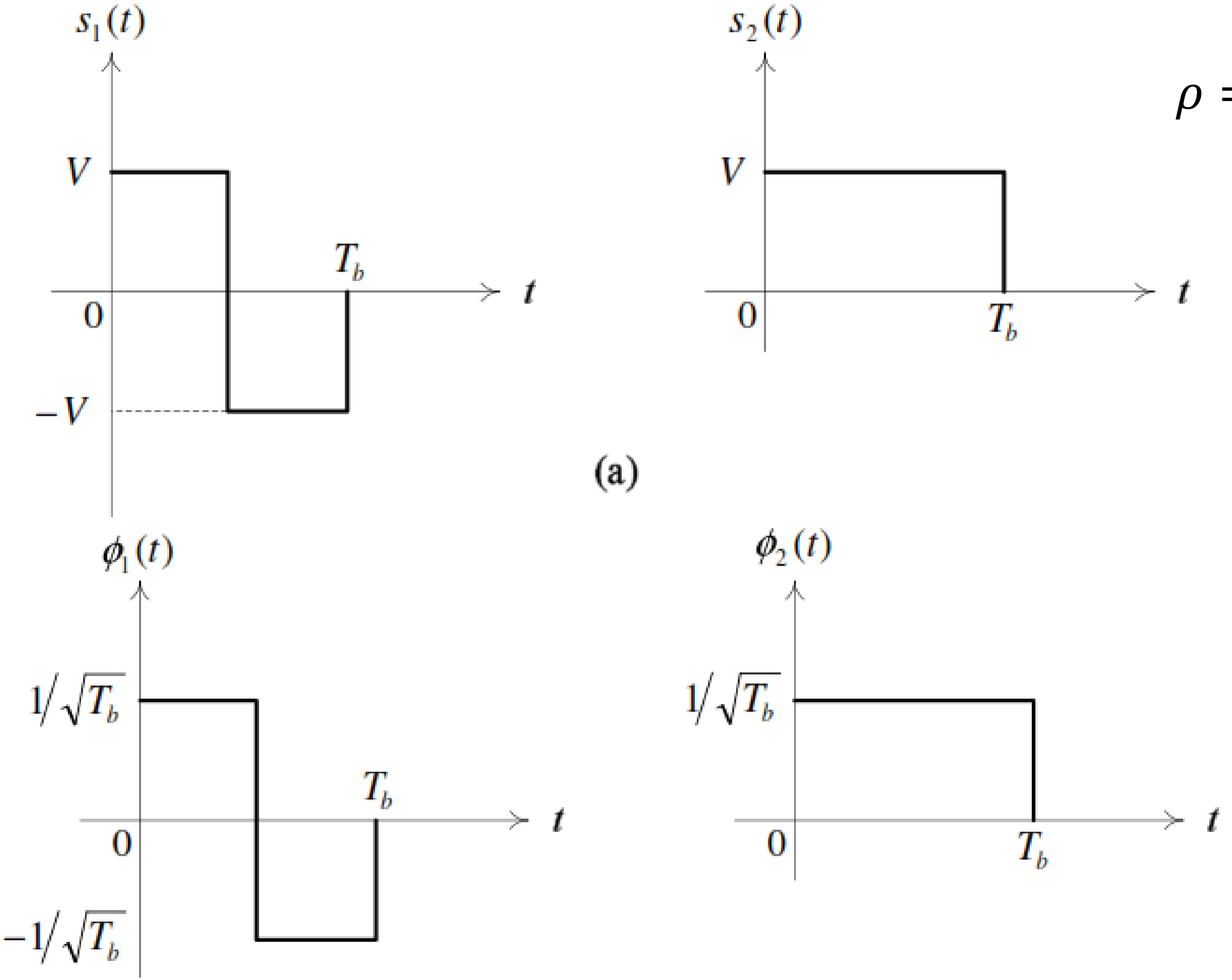
(a) Signal set. (b) Orthonormal function. (c) Signal space representation.

For the case of binary antipodal signaling, i.e., when $s_2(t) = -s_1(t)$, we need only one basis function. The signals are represented as:

$$s_1(t) = \sqrt{E}\phi_1(t)$$

$$s_2(t) = -\sqrt{E}\phi_1(t)$$

Example: Orthogonal Binary Signals



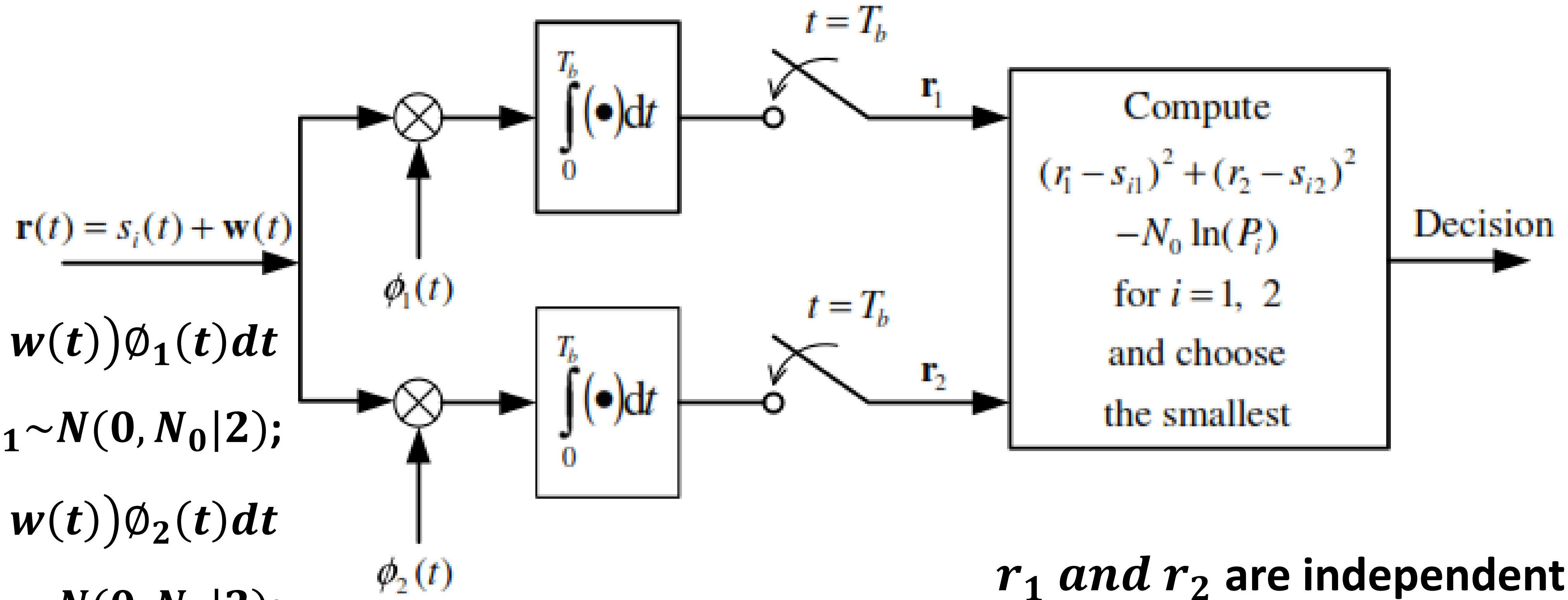
$$\rho = \int_0^{T_b} s_1(t)s_2(t)dt = 0$$

For the case of binary signaling when ρ =0 we need two bases functions. The signals are represented as:

$$\boldsymbol{s_1(t) = \sqrt{E}\phi_1(t)}$$

$$\boldsymbol{s_2(t) = \sqrt{E}\phi_2(t)}$$

Optimum Receiver : Matched Filter and Correlators



$$r_1 = \int_0^{T_b} (s_i(t) + w(t))\emptyset_1(t)dt$$

$$r_1 = s_{i1} + N_1; N_1 \sim N(0, N_0|2);$$

$$r_2 = \int_0^{T_b} (s_i(t) + w(t))\emptyset_2(t)dt$$

$$r_1 = s_{i2} + N_2; N_2 \sim N(0, N_0|2);$$

r_1 *and* r_2 are independent

$r_1 \sim N(s_{i1}, N_0|2)$; Gaussian with mean s_{i1}, variance $N_0|2$

$r_2 \sim N(s_{i2}, N_0|2)$; Gaussian with mean s_{i2}, variance $N_0|2$

The receiver computes the coordinates of the received signal in the $\phi_1(t) - \phi_1(t)$ plane and makes a decision according to the closeness of these coordinates from those of the transmitted signals as illustrated on the next slide.

The likelihood Ratio Test

- The probability of error is minimized when the following decision rule is employed:
- Decide $\widehat{b_i} = 1 \ when \ \frac{\boldsymbol{f(r_1,r_2|b_i=1)}}{\boldsymbol{f(r_1,r_2|b_i=0)}} \geq \frac{\boldsymbol{P_2}}{\boldsymbol{P_1}};$ (1)
- Decide $\widehat{b_i} = 0 \ when \ \frac{\boldsymbol{f(r_1,r_2|b_i=1)}}{\boldsymbol{f(r_1,r_2|b_i=0)}} < \frac{\boldsymbol{P_2}}{\boldsymbol{P_1}};$
- $f(r_1, r_2|b_i = 1) = f(r_1 \, |b_i = 1)f(r_2|b_i = 1)$; due to independence
- $f(r_1, r_2|b_i = 0) = f(r_1 \, |b_i = 0)f(r_2|b_i = 0)$; due to independence
- $f(r_1 \, |b_i = 1) \sim N\left(s_{11}, \frac{N_0}{2}\right); f(r_2 \, |b_i = 1) \sim N\left(s_{12}, \frac{N_0}{2}\right);$
- $f(r_1 \, |b_i = 0) \sim N\left(s_{21}, \frac{N_0}{2}\right); f(r_2 \, |b_i = 0) \sim N\left(s_{22}, \frac{N_0}{2}\right);$
- Substituting into (1), and simplifying, we get

Optimum Receiver: Binary Case

- Simplified decision rule when the noise $\mathbf{w}(t)$ is zero-mean, white and Gaussian:

$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 + (r_2 - s_{22})^2 + N_0 \ln\left(\frac{P_1}{P_2}\right)$$

- For the special case of $P_1 = P_2$ (signals are equally likely):

$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 + (r_2 - s_{22})^2.$$

The Minimum Distance Decision Rule

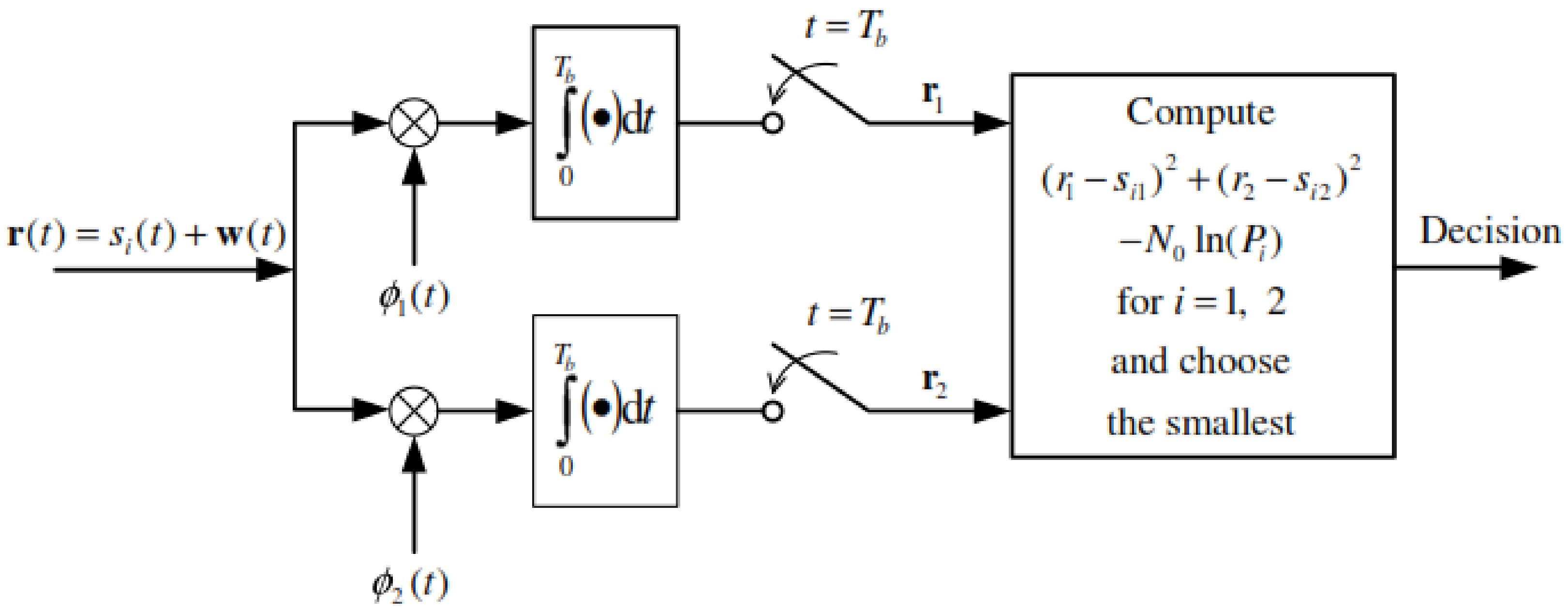
The probability of error for Equally-probable signals is given by:

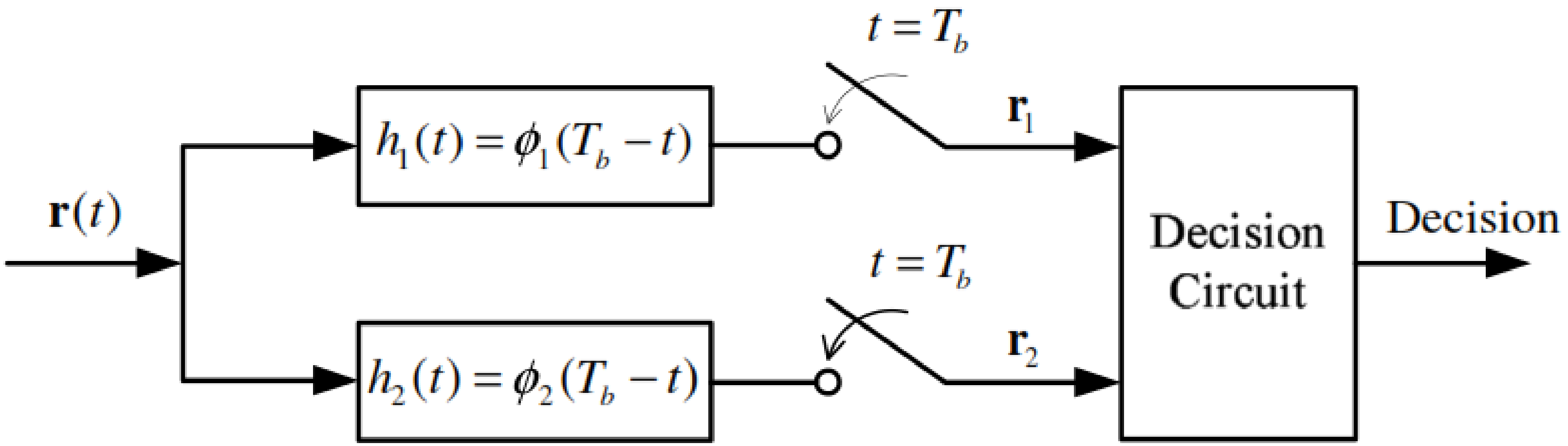
$$P_b^* = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{\int_0^{T_b}\left(s_1(t) - s_2(t)\right)^2 dt}{2N_0}}\right)$$

As derived earlier

$$d_{12}^2 = \int_0^{T_b} (s_1(t) - s_2(t))^2 dt = (s_{11} - s_{21})^2 + (s_{12} - s_{22})^2$$

Optimum Receiver : Matched Filter and Correlators





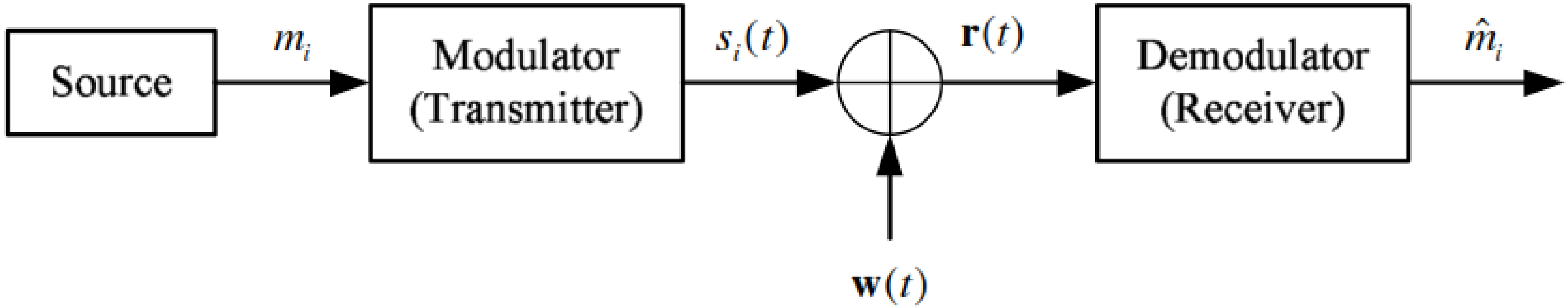
The receiver can be implemented in terms of correlators and can, as well, be implemented in terms of the matched filters. Here, matched means that the filters at the receiver are matched to the bases functions used in the transmission process. The two figures on this slide are equivalent in terms of performance.

M-Ary Transmission

- There are benefits to be gained when M-ary $(M = 4)$ signaling methods are used rather than straightforward binary signaling.
- In general, M-ary communication is used when one needs to design a communication system that is bandwidth efficient.
- Unlike QPSK and its variations, the gain in bandwidth is accomplished at the expense of error performance.
- To use M-ary modulation, the bit stream is blocked into groups of λ bits $\Rightarrow$ the number of bit patterns is $M = 2^{\lambda}$.
- The symbol transmission rate is $r_s = 1/T_s = 1/(\lambda T_b) = r_b/\lambda$ symbols/sec $\Rightarrow$ there is a bandwidth saving of $1/\lambda$ compared to binary modulation.
- Shall consider M-ary ASK, PSK, QAM (quadrature amplitude modulation) and FSK.

For each modulation scheme, we will consider the transmitter, the optimum receiver, the probability of error, the power spectral density and the bandwidth

Optimum Receiver for M-Ary Transmission



- $\mathbf{w}(t)$ is zero-mean white Gaussian noise with power spectral density of $\frac{N_0}{2}$ (watts/Hz).
- Receiver needs to make the decision on the transmitted signal based on the received signal $\mathbf{r}(t) = s_i(t) + \mathbf{w}(t)$.
- The determination of the optimum receiver (with minimum error) proceeds in a manner analogous to that for the binary case.

Optimum Receiver for M-Ary Transmission

- Represent M signals by an orthonormal basis set, $\{\phi_n(t)\}_{n=1}^{N}$, $N \leq M$:

$$\begin{aligned} s_i(t) &= s_{i1}\phi_1(t) + s_{i2}\phi_2(t) + \cdots + s_{iN}\phi_N(t), \\ s_{ik} &= \int_0^{T_s} s_i(t)\phi_k(t)\mathrm{d}t. \end{aligned}$$

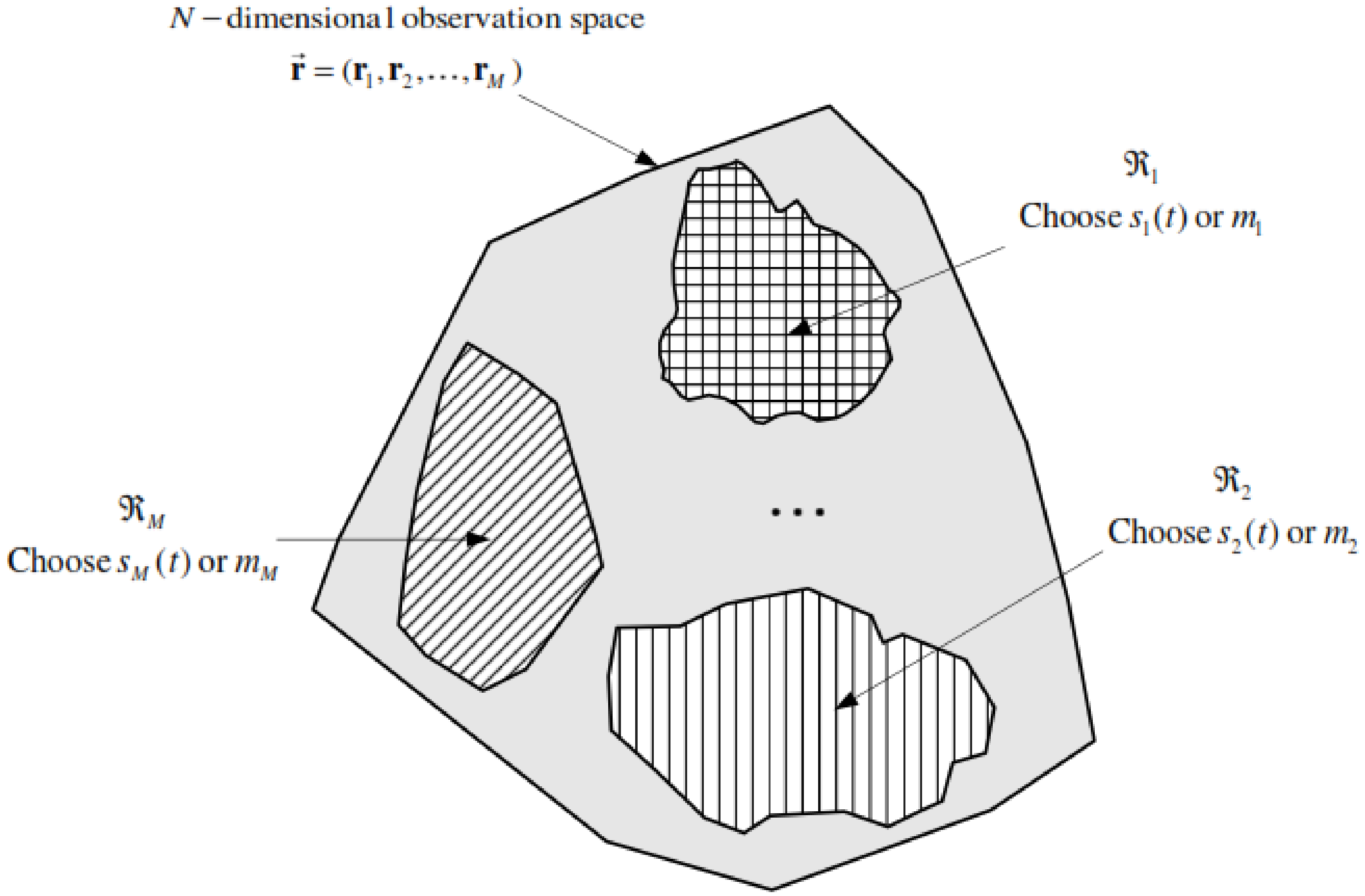
Here, we are given M signals. However, there are $N \leq M$ bases functions which can be obtained using the Gram Schmidt orthogonalization procedure. Each one of the M signals, can be represented as a linear combination of the N bases functions

- Expand the received signal $\mathbf{r}(t)$ into the series

$$\begin{aligned} \mathbf{r}(t) &= s_i(t) + \mathbf{w}(t) \\ &= \mathbf{r}_1\phi_1(t) + \mathbf{r}_2\phi_2(t) + \cdots + \mathbf{r}_N\phi_N(t) + \mathbf{r}_{N+1}\phi_{N+1}(t) + \cdots \end{aligned}$$

- For $k > N$, the coefficients $\mathbf{r}_k$ can be discarded.
- Need to partition the N-dimensional space formed by $\vec{\mathbf{r}} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ into M regions so that the message error probability is minimized.

Optimum Receiver for M-Ary Transmission



The optimum receiver is also the *minimum-distance receiver*:

Choose m_i if
$$\sum_{k=1}^{N}(r_k - s_{ik})^2 < \sum_{k=1}^{N}(r_k - s_{jk})^2;$$
$$j = 1, 2, \ldots, M;\ j \neq i.$$

The observation space is to be partitioned into M regions, such that if the set of measurements fall into region R_k signal s_k is declared true.

It is assumed here that all signals are equally probable.

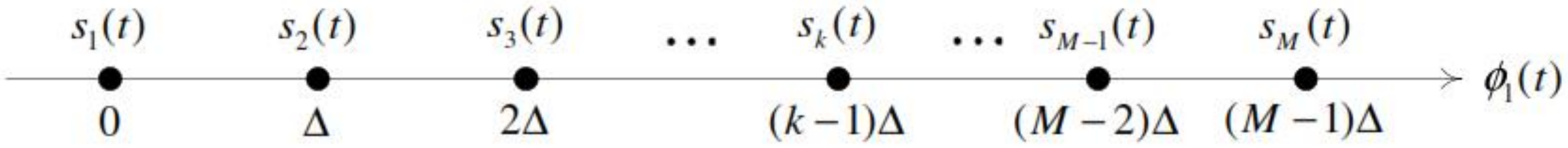
The receiver collects the measurements from the N correlators (r vector) and calculates the distance to each of the N signals.

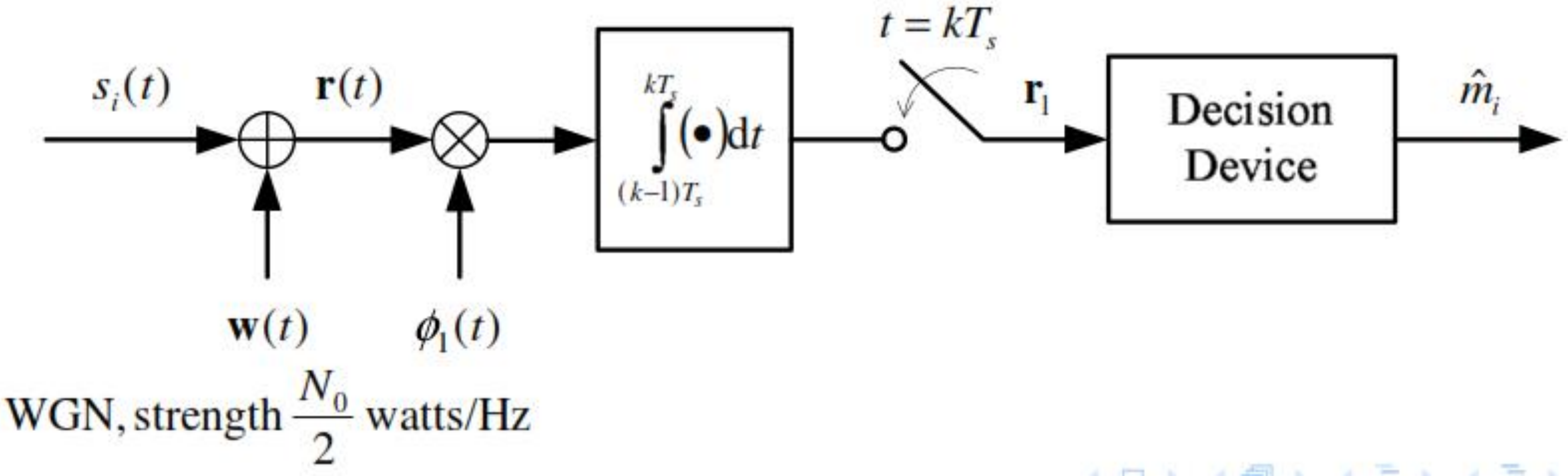
It decides in favor of the signal closest to the (r vector).

M-ary Coherent Amplitude-Shift Keying (M-ASK)

$$
\begin{aligned}
s_i(t) &= V_i\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t),\ 0 \le t \le T_s \\
&= [(i-1)\Delta]\phi_1(t), \quad \phi_1(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t),\, 0 \le t \le T_s, \\
& \quad i = 1, 2, \ldots, M.
\end{aligned}
$$

In this case, we have M signals. However, we need only one base function. Every signal can be expressed in terms of this base function.

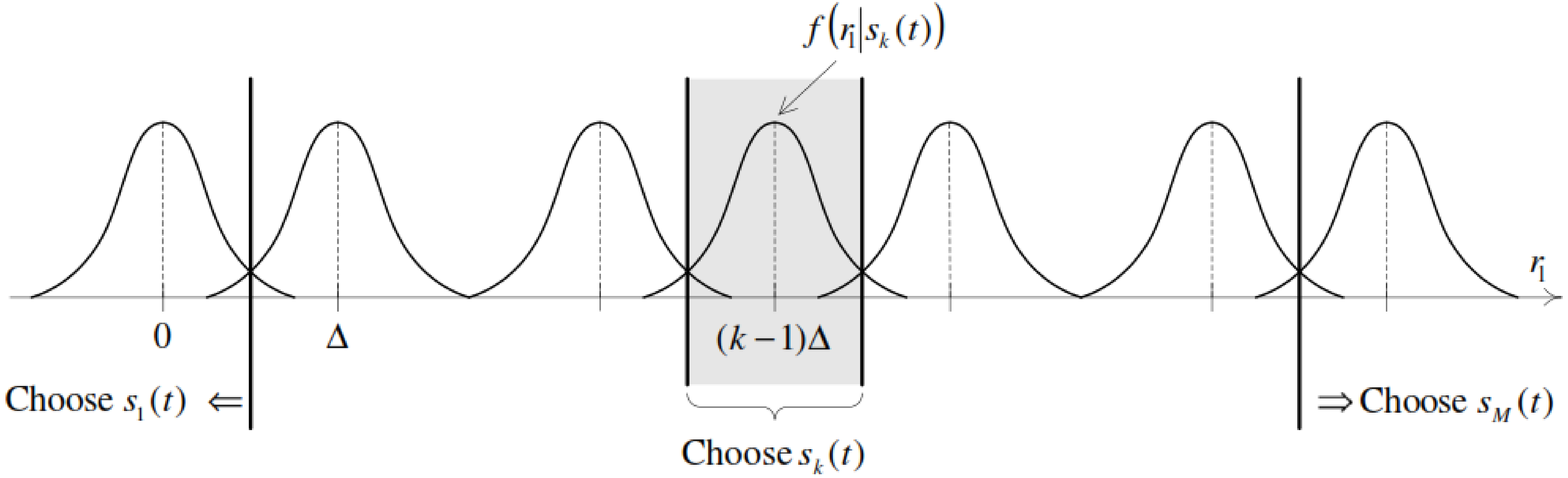




The receiver consists one correlator (multiplier followed by an integrator), a sampler, and a decision device (set of comparators).

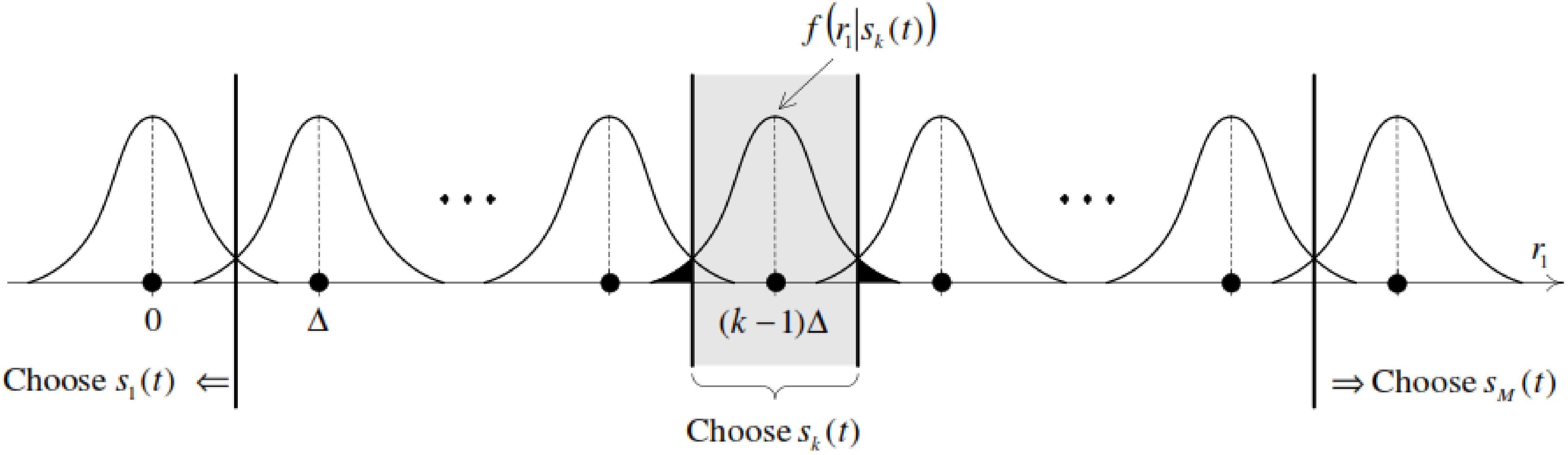
Minimum-Distance Decision Rule for M-ASK

$$\text{Choose} \begin{cases} s_k(t), & \text{if} \quad \left(k-\frac{3}{2}\right)\Delta < r_1 < \left(k-\frac{1}{2}\right)\Delta, \ k = 2,3,\ldots,M-1 \\ s_1(t), & \text{if} \quad r_1 < \frac{\Delta}{2} \\ s_M(t), & \text{if} \quad r_1 > \left(M-\frac{3}{2}\right)\Delta \end{cases}$$



Minimum-Distance Decision Rule for M-ASK

$$\text{Choose} \begin{cases} s_k(t), & \text{if} \quad \left(k-\frac{3}{2}\right)\Delta < r_1 < \left(k-\frac{1}{2}\right)\Delta, \; k = 2,3,\ldots,M-1 \\ s_1(t), & \text{if} \quad r_1 < \frac{\Delta}{2} \\ s_M(t), & \text{if} \quad r_1 > \left(M-\frac{3}{2}\right)\Delta \end{cases} .$$



$$P[\text{error}] = \sum_{i=1}^{M} P[s_i(t)] P[\text{error}|s_i(t)]$$

$$P[\text{error}|s_i(t)] = 2Q\left(\Delta/\sqrt{2N_0}\right), \quad i = 2,3,\ldots,M-1$$

$$P[\text{error}|s_i(t)] = Q\left(\Delta/\sqrt{2N_0}\right), \quad i = 1, M$$

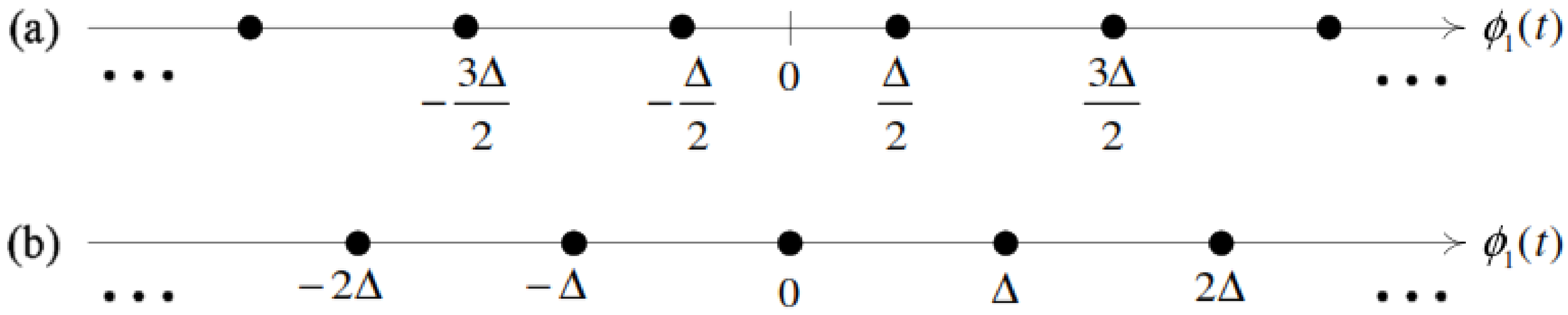
$$P[\text{error}] = \frac{2(M-1)}{M} Q\left(\Delta/\sqrt{2N_0}\right).$$

For a given M, $P[\text{error}]$ depends on the noise power (N_0) and the *minimum* distance δ. This means that moving the origin of the signal constellation does not affect the performance!

Modified M-ASK Constellation

The maximum and average transmitted energies can be reduced, without any sacrifice in error probability, by changing the signal set to one which includes the negative version of each signal.

$$s_i(t) = \underbrace{(2i-1-M)\frac{\Delta}{2}}_{V_i}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t),\ 0 \le t \le T_s,\ i = 1, 2, \ldots, M.$$



$$E_s = \frac{\sum_{i=1}^{M} E_i}{M} = \frac{\Delta^2}{4M}\sum_{i=1}^{M}(2i-1-M)^2 = \frac{(M^2-1)\Delta^2}{12}.$$

Es: Average Energy per Symbol

$$E_b = \frac{E_s}{\log_2 M} = \frac{(M^2-1)\Delta^2}{12\log_2 M} \Rightarrow \Delta = \sqrt{\frac{(12\log_2 M)E_b}{M^2-1}}$$

Eb: Average Energy per bit

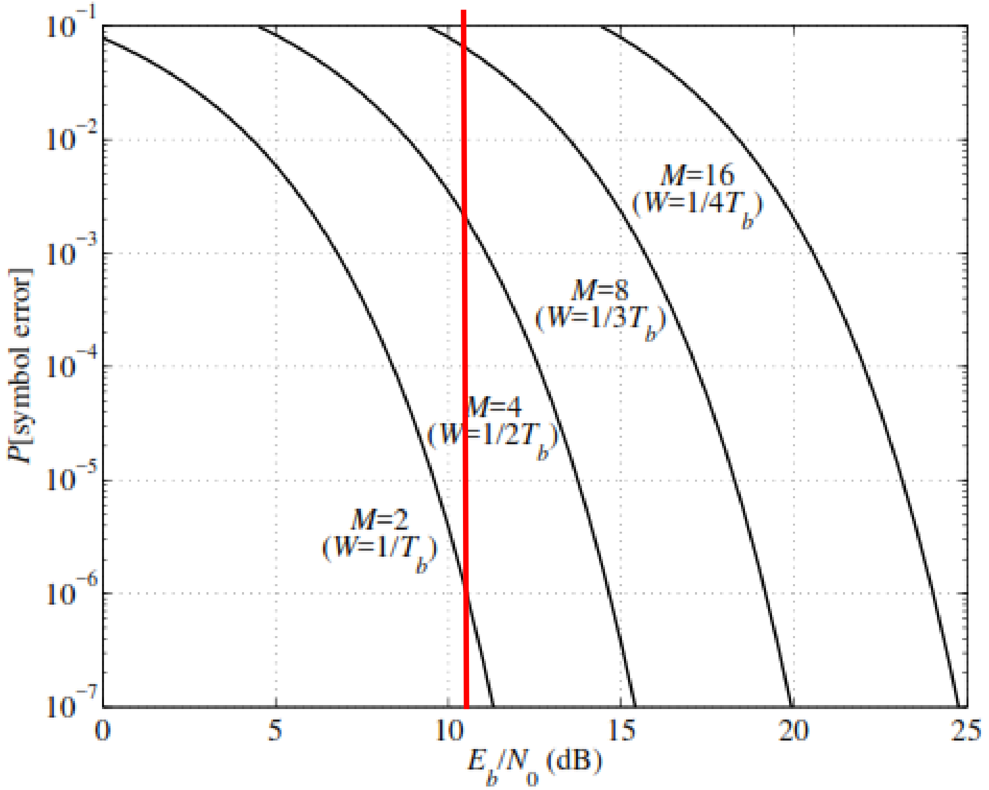
Probability of Symbol Error for M-ASK

$$P[\text{error}] = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6E_s}{(M^2-1)N_0}}\right) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right).$$

Symbol error probability

$$P[\text{bit error}] = \frac{1}{\lambda} P[\text{symbol error}] = \frac{2(M-1)}{M\log_2 M} Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right) \quad \text{(with Gray mapping)}$$

Bit error probability



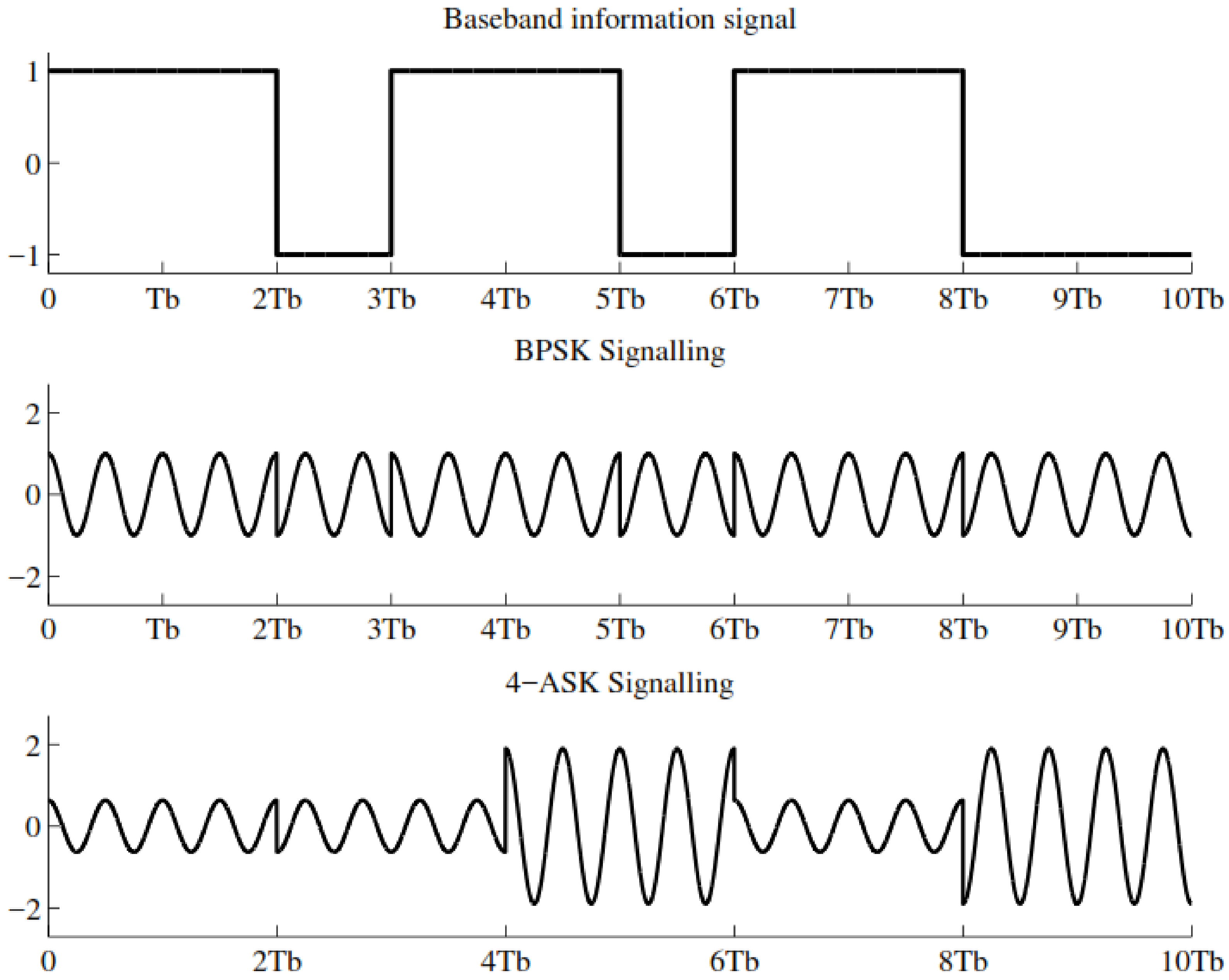
Two comments:

Error probability: for a given Eb/N0, increasing M results in an increase in the error probability.

Bandwidth: Increasing M results in a reduction in the bandwidth by a factor of $\lambda = log_2(M)$.

W is obtained by using the $WT_s = 1$ rule-of-thumb. Here $1/T_b$ is the bit rate (bits/s).

Example of 2-ASK (BPSK) and 4-ASK Signals



Binary sequence:
1101101100

$1 \rightarrow cos(2\pi f_0)\, t$
$0 \rightarrow -cos(2\pi f_0)\, t$
(similar to BPSK)

$11 \rightarrow \cos(2\pi f_0)\, t$
$01 \rightarrow -cos(2\pi f_0)\, t$
$10 \rightarrow 2cos(2\pi f_0)\, t$
$00 \rightarrow -2\, cos(2\pi f_0)\, t$

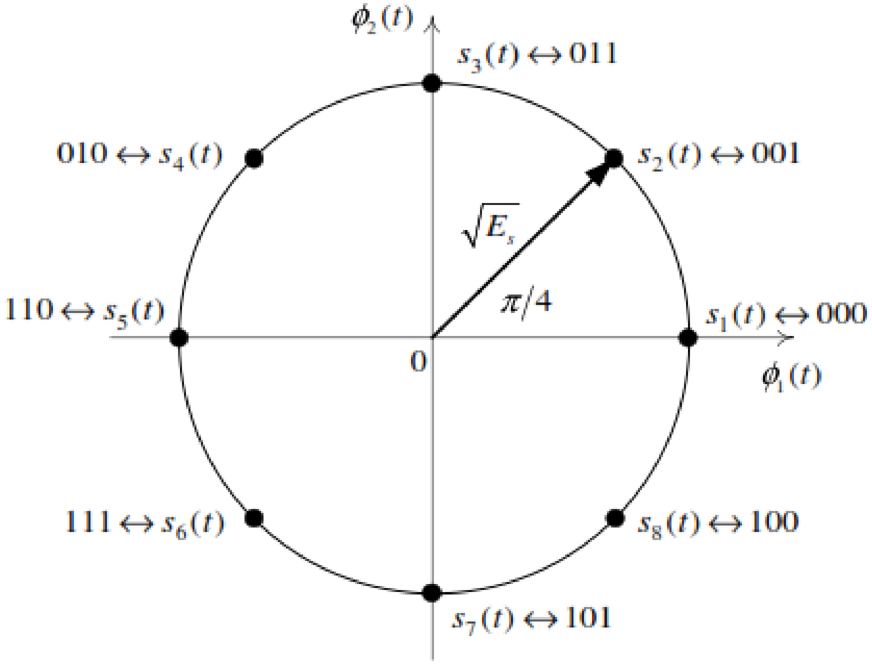
M-ary Phase-Shift Keying (M-PSK)

$$s_i(t) = V\cos\left[2\pi f_c t - \frac{(i-1)2\pi}{M}\right],\ 0 \le t \le T_s;\ i = 1, 2, \ldots, M;\ E_s = V^2T_s/2$$

$$= V\cos\left[\frac{(i-1)2\pi}{M}\right]\cos(2\pi f_c t) + V\sin\left[\frac{(i-1)2\pi}{M}\right]\sin(2\pi f_c t).$$

$$s_{i1} = \sqrt{E_s}\cos\left[\frac{(i-1)2\pi}{M}\right],\ s_{i2} = \sqrt{E_s}\sin\left[\frac{(i-1)2\pi}{M}\right].$$

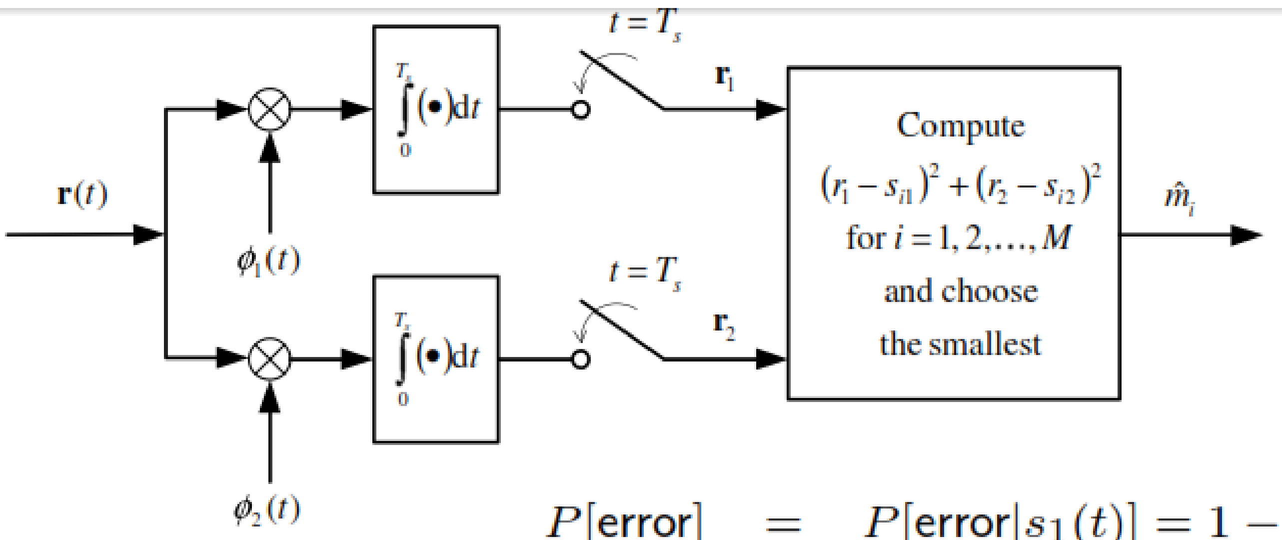
Signal Energy $E_s = V^2T_s/2$



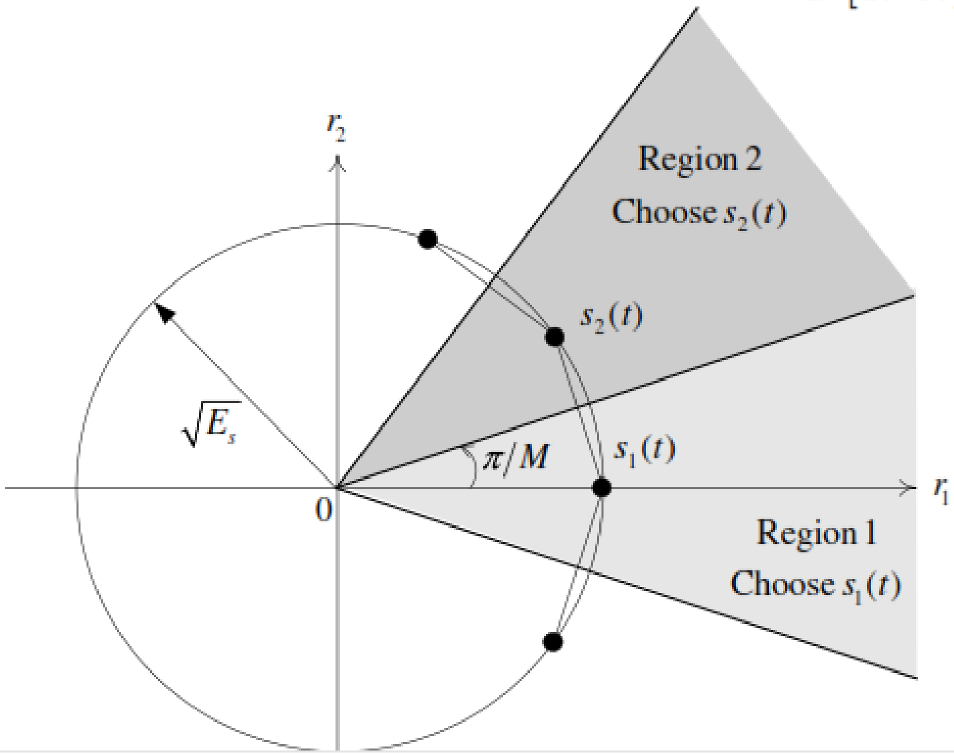
Here, the amplitude of the carrier remains constant, however the phase takes on one of M possible values.
Two base functions are needed to represent all signals in the two-dimensional signal space.
The spacing between adjacent signals is $\Delta\theta = 2\pi/M$ radians.
In this example, M=8 and $\Delta\theta = \frac{\pi}{4} = 45\ degrees$.
To minimize error, gray coding is used.

The signals lie on a circle of radius $\sqrt{E_s}$, and are spaced every $2\pi/M$ radians around the circle.

Optimum Receiver for M-PSK



Compute d_i^2, for all i and choose the signal corresponding to the smallest signal.



$$P[\text{error}] = P[\text{error}|s_1(t)] = 1 - P[\text{correct}|s_1(t)]$$

$$= 1 - \underbrace{\iint_{r_1,r_2\in\text{Region 1}} f(r_1,r_2|s_1(t))\mathrm{d}r_1\mathrm{d}r_2}_{P[\text{correct}|s_1(t)]}.$$

Need two correlators, since we have two base functions. The two correlators compute the coordinates of the received signal in the two-dimensional space. The minimum distance rule is employed. The signal with the smallest distance to the received (r1, r2) coordinates is selected. The shaded regions in the figure specify the decision region for each signal.

Probability of Error in M-PSK

- The distance between two neighboring symbols is

$$d_{\min} = 2\sqrt{E_s}\sin\left(\frac{\pi}{M}\right)$$

- Each symbol has 2 neighbor symbols.

- An approximation for the probability of symbol error is

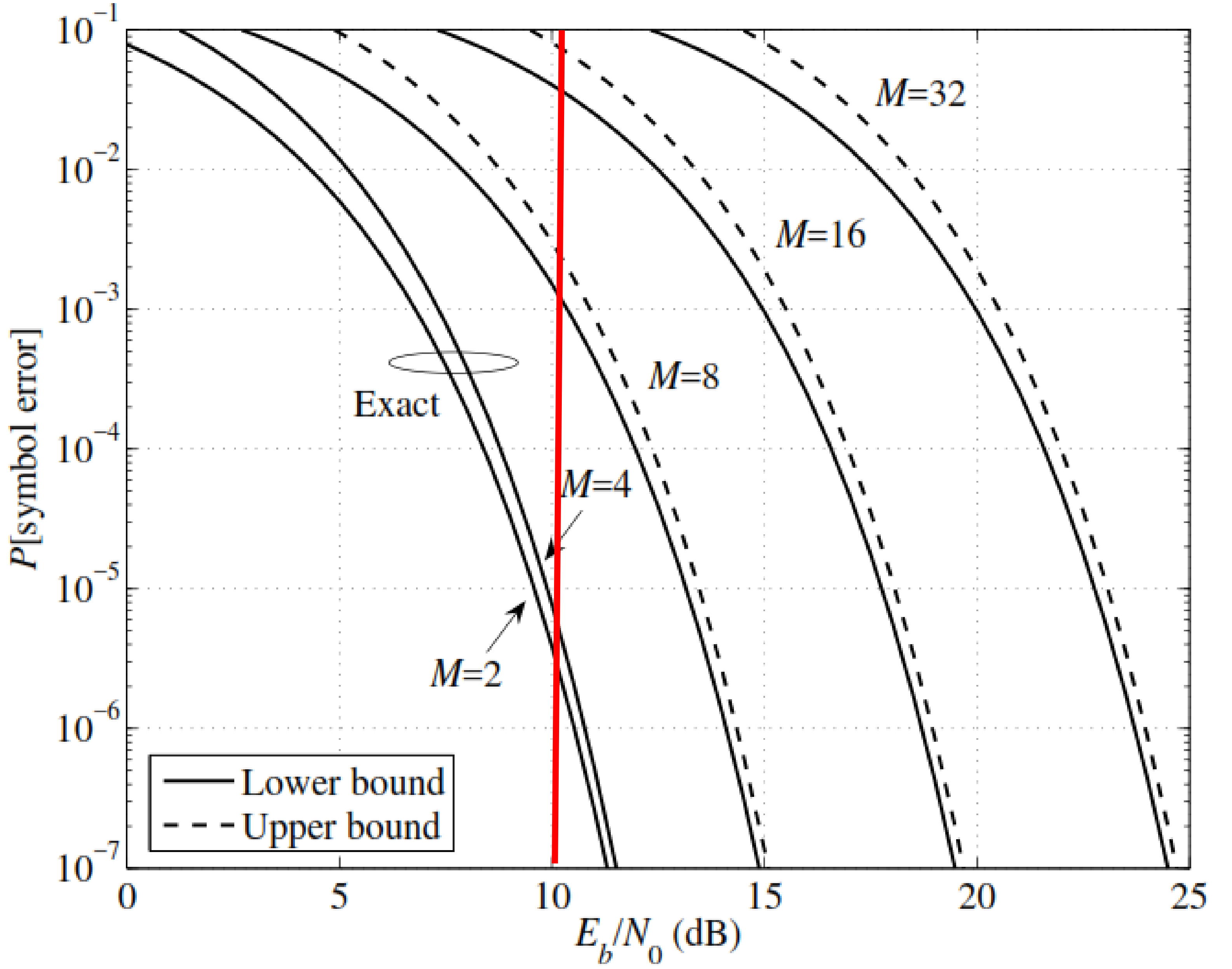
$$P_s \approx 2 \cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 2 \cdot Q\left(\frac{2\sqrt{E_s}\sin\left(\frac{\pi}{M}\right)}{\sqrt{2N_0}}\right) = 2 \cdot Q\left(\sqrt{2\frac{E_s}{N_0}\sin^2\left(\frac{\pi}{M}\right)}\right)$$

When M=4, we have QPSK. The symbol error probability is: $P_s^{QPSK} \approx 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

Symbol and Bit Error Probability of M-PSK

- When Gray coding is used, the symbol and bit error probabilities are related by: $\boldsymbol{P_b = \frac{1}{log_2(M)} P_S}$;
- Moreover, the symbol energy is related to the bit energy by
- $\boldsymbol{E_b = \frac{1}{log_2(M)} E_S}$
- The performance of digital communication systems is usually taken as the error probability versus $\frac{E_b}{N_0}$.
- The next figure depicts the symbol probability of error for M-PSK

Performance of M-PSK



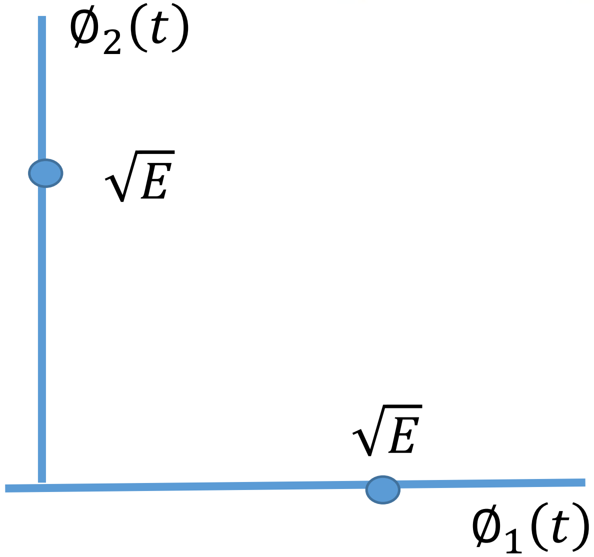
As M increases, the symbol probability of error increases. Note that as M increases, the spacing between signals around the perimeter of the unit circle becomes smaller, and this results in a higher probability of error

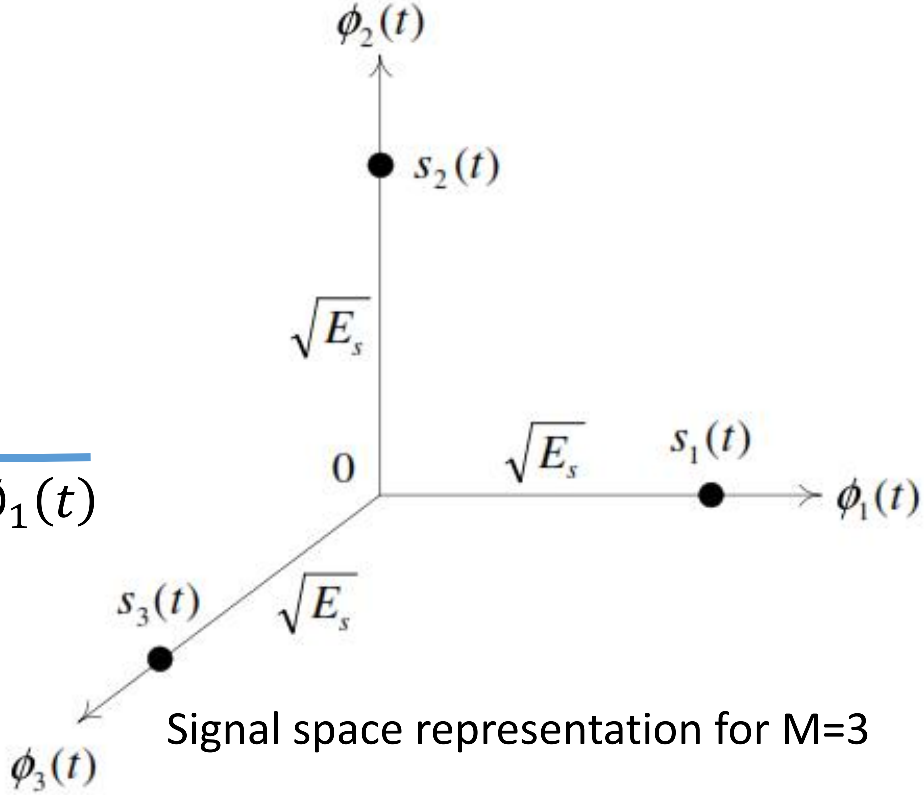
M-ary Coherent Frequency-Shift Keying (M-FSK)

$$s_i(t) = \begin{cases} V\cos(2\pi f_i t), & 0 \le t \le T_s \\ 0, & \text{elsewhere} \end{cases}, \quad i = 1, 2, \ldots, M,$$

where f_i are chosen to have orthogonal signals over $[0, T_s]$.

$$f_i = \begin{cases} (k \pm i)\left(\frac{1}{2T_s}\right), & \text{(coherently orthogonal)} \\ (k \pm i)\left(\frac{1}{T_s}\right), & \text{(noncoherently orthogonal)} \end{cases}, \quad i = 0, 1, 2, \ldots$$





Signal space representation for M=3

Orthogonality condition:

$$\int_0^{T_s} s_i(t)s_j(t)\,dt = 0, i \neq j$$

All signals have the same energy

$$E = \int_0^{T_s} s_i(t)^2\ dt = \frac{V^2 T_s}{2}$$

As a result of this condition, there will be M base functions

$$\phi_i(t) = \frac{s_i(t)}{\sqrt{E}} = \sqrt{\frac{2}{T_s}}\cos(2\pi f_i t$$

Minimum-Distance Receiver of M-FSK

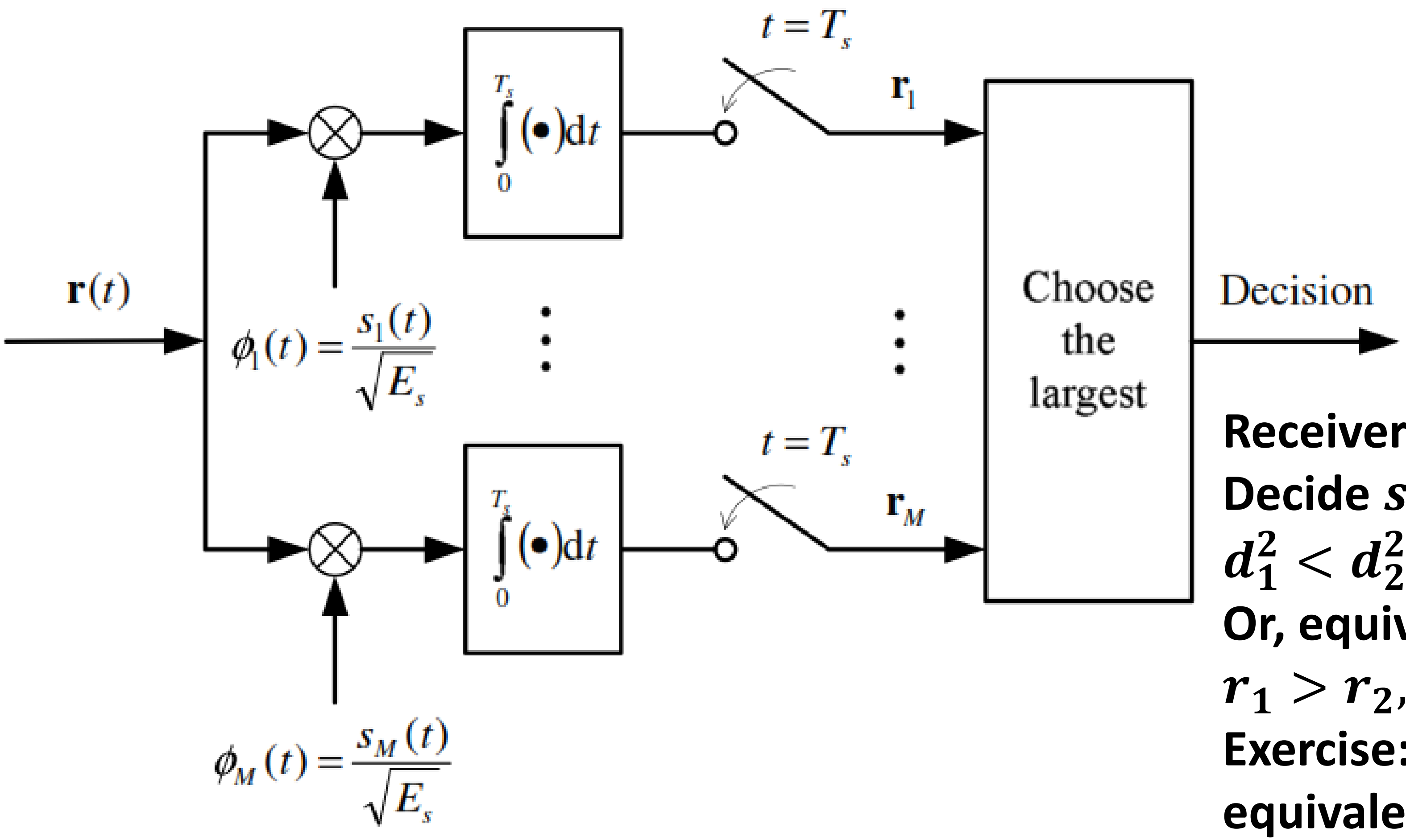
Choose m_i if

$$\sum_{k=1}^{M}(r_k - s_{ik})^2 < \sum_{k=1}^{M}(r_k - s_{jk})^2 \quad \Rightarrow$$

$j = 1, 2, \ldots, M;\ j \neq i,$

Choose m_i if

$r_i > r_j, \quad j = 1, 2, \ldots, M;\ j \neq i.$



The receiver consists of M correlators (corresponding to the M base functions) followed by the decision maker. The decision maker employs the minimum distance rule.

Receiver computes $d_1^2, d_2^2, \ldots, d_M^2$

Decide s_1 when

$d_1^2 < d_2^2, d_1^2 < d_3^2, \ldots, d_1^2 < d_M^2$

Or, equivalently when

$r_1 > r_2, r_1 > r_3, \ldots, r_1 > r_M$

Exercise: Prove the latter equivalency condition

Union Bound on the Symbol Error Probability of M-FSK

$$P[\text{error}] = P[(\mathbf{r}_1 < \mathbf{r}_2) \text{ or } (\mathbf{r}_1 < \mathbf{r}_3) \text{ or}, \cdots, \text{or } (\mathbf{r}_1 < \mathbf{r}_M) | s_1(t)].$$

- Since the events are not mutually exclusive, the error probability is bounded by:

$$P[\text{error}] < P[(\mathbf{r}_1 < \mathbf{r}_2)|s_1(t)] + P[(\mathbf{r}_1 < \mathbf{r}_3)|s_1(t)] + \cdots + P[(\mathbf{r}_1 < \mathbf{r}_M)|s_1(t)].$$

The error bound is tight for high signal to noise ratio.

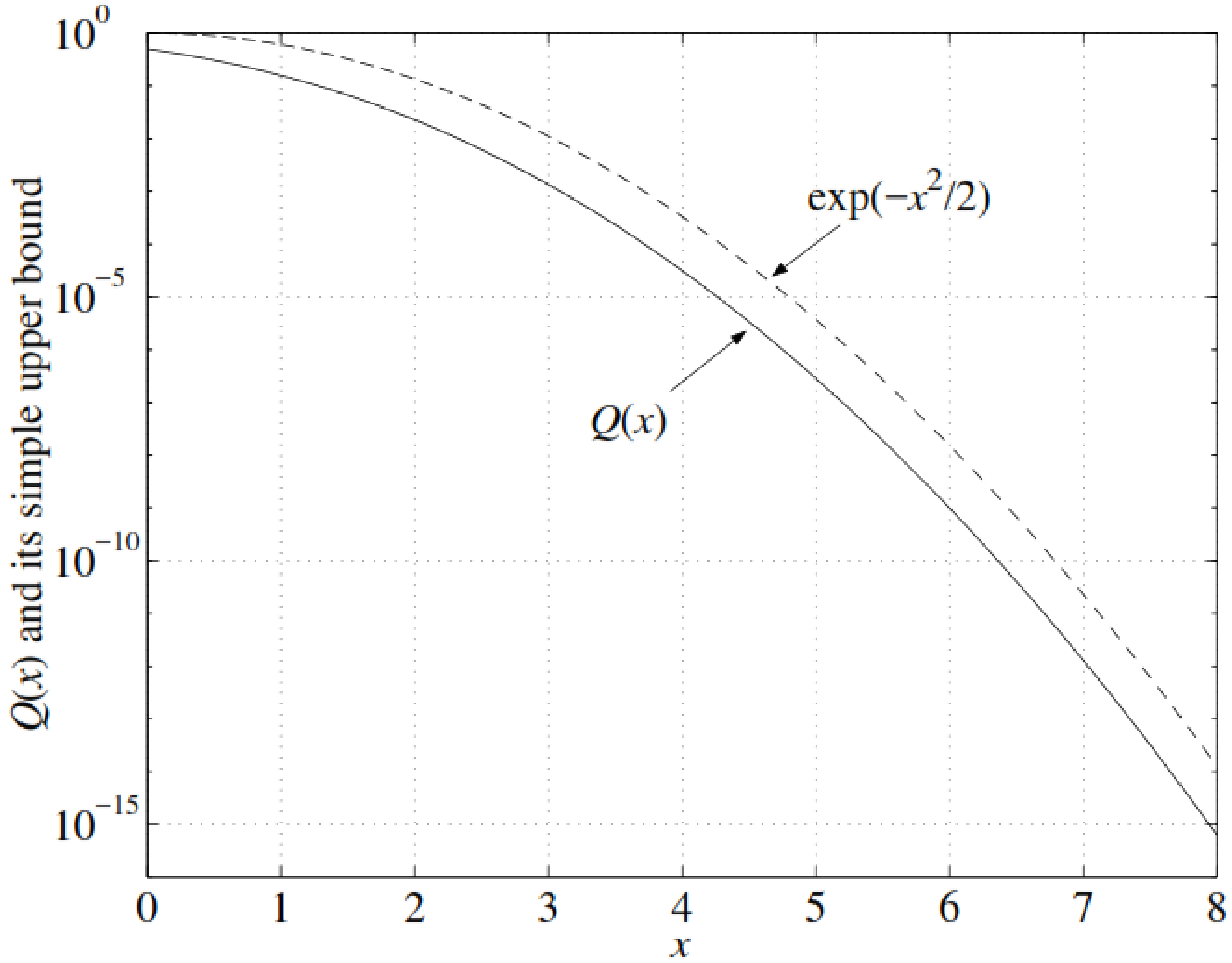
- But $P[(\mathbf{r}_1 < \mathbf{r}_j)|s_1(t)] = Q\left(\sqrt{E_s/N_0}\right)$, $j = 3, 4, \ldots, M$. Then

$$P[\text{error}] < (M-1)Q\left(\sqrt{E_s/N_0}\right) < MQ\left(\sqrt{E_s/N_0}\right) < M\mathrm{e}^{-E_s/(2N_0)}.$$

where the bound $Q(x) < \exp\left\{-\frac{x^2}{2}\right\}$ has been used.

An Upper Bound on Q(x)

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\lambda^2}{2}\right\} d\lambda < \exp\left\{-\frac{x^2}{2}\right\}$$



Interpretations of P[error] < $Me^{-E_s/(2N_0)}$

1. Let $M = 2^\lambda = e^{\lambda \ln 2}$ and $E_s = \lambda E_b$. Then

$$P[\text{error}] < e^{\lambda \ln 2} e^{-\lambda E_b/(2N_0)} = e^{-\lambda(E_b/N_0 - 2\ln 2)/2}.$$

As $\lambda \to \infty$, or equivalently, as $M \to \infty$, the probability of error *approaches zero* exponentially, provided that

$$\frac{E_b}{N_0} > 2\ln 2 = 1.39 = 1.42 \text{ dB}.$$

2. Since $E_s = \lambda E_b = V^2 T_s/2$, then

$$P[\text{error}] < e^{\lambda \ln 2} e^{-V^2 T_s/(4N_0)} = e^{-T_s[-r_b \ln 2 + V^2/(4N_0)]}$$

If $-r_b \ln 2 + V^2/(4N_0) > 0$, or $r_b < \frac{V^2}{4N_0 \ln 2}$ the probability or error tends to zero as T_s or M becomes larger and larger.

Unlike M-ASK and M-PSK, this slide shows that the symbol error probability decreases as M increases.

Rb: is the data rate below which the probability of error can be made arbitrarily small using M-FSK

This results shows that it is possible to make the probability of error arbitrarily small even for a fixed signal to noise ratio.

Symbol Error Probability of M-FSK

$$P[\text{error}] = P[\text{error}|s_1(t)] = 1 - P[\text{correct}|s_1(t)].$$

$$P[\text{correct}|s_1(t)] \quad = \quad P[(\mathbf{r}_2 < \mathbf{r}_1) \text{ and } \cdots \text{ and } (\mathbf{r}_M < \mathbf{r}_1)|s_1(t) \text{ sent}].$$

$$= \int_{r_1=-\infty}^{\infty} P[(\mathbf{r}_2 < r_1) \text{ and} \cdots \text{and } (\mathbf{r}_M < r_1)|\{\mathbf{r}_1 = r_1, s_1(t)\}] f(r_1|s_1(t))\mathrm{d}r.$$

$$P[(\mathbf{r}_2 < r_1) \text{ and} \cdots \text{and}(\mathbf{r}_M < r_1)|\{\mathbf{r}_1 = r_1, s_1(t)\}] = \prod_{j=2}^{M} P[(\mathbf{r}_j < r_1)|\{\mathbf{r}_1 = r_1, s_1(t)\}].$$

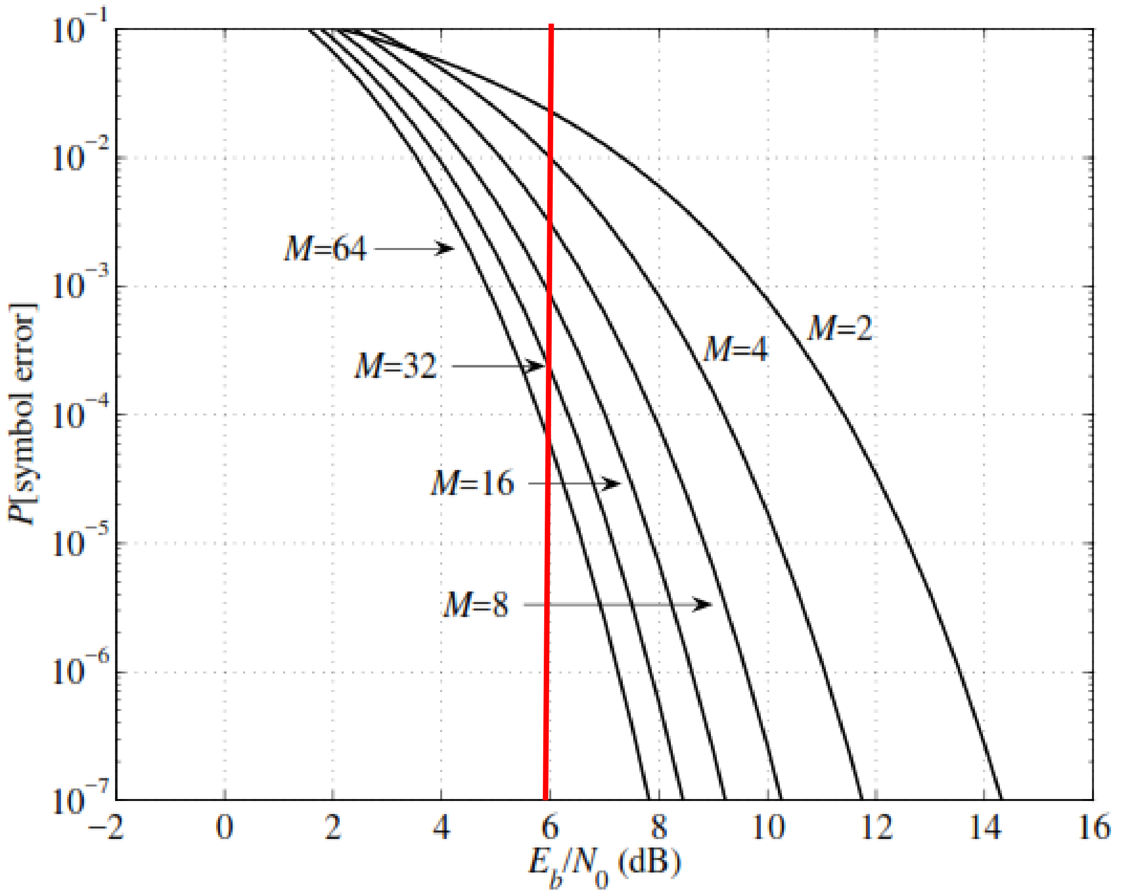
$$P[\mathbf{r}_j < r_1|\{\mathbf{r}_1 = r_1, s_1(t)\}] = \int_{-\infty}^{r_1} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{\lambda^2}{N_0}\right\} \mathrm{d}\lambda.$$

$$P[\text{correct}] \quad = \quad \int_{r_1=-\infty}^{\infty} \left[\int_{\lambda=-\infty}^{r_1} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{\lambda^2}{N_0}\right\} \mathrm{d}\lambda\right]^{M-1} \times$$

$$\frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(r_1 - \sqrt{E_s})^2}{N_0}\right\} \mathrm{d}r_1.$$

Symbol Error Probability of M-FSK

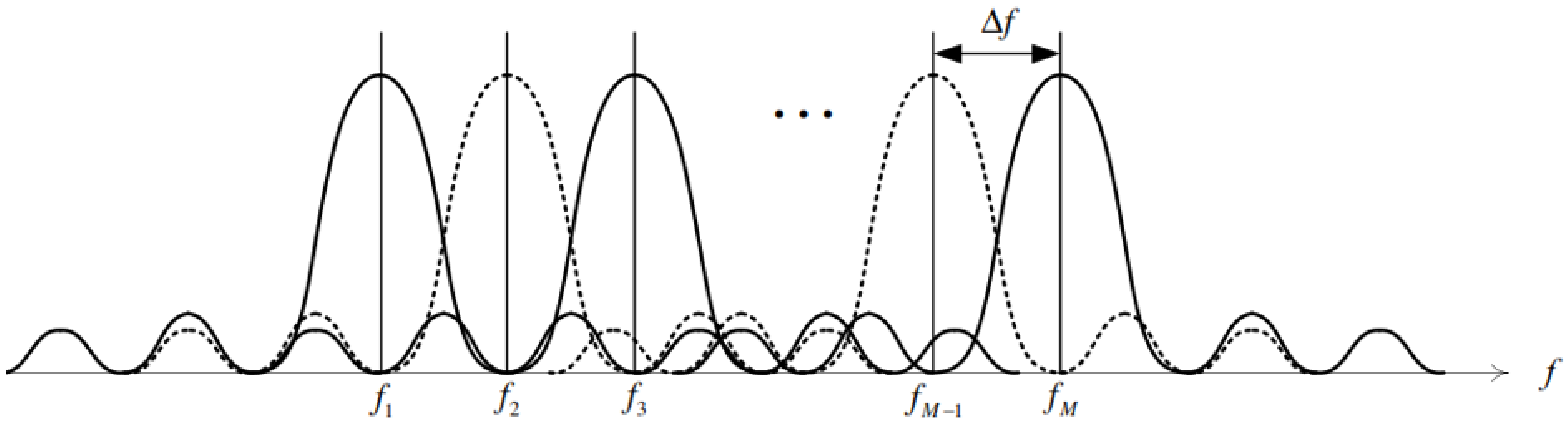
$$P[\text{error}] = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \mathrm{e}^{-x^2/2} \mathrm{d}x \right]^{M-1} \exp\left[-\frac{1}{2} \left(y - \sqrt{\frac{2\log_2 M E_b}{N_0}} \right)^2 \right] \mathrm{d}y.$$



With M-FSK, the required E_b/N_0 to achieve a given error probability decreases as M increases. Since increasing M means increasing transmission bandwidth. **Thus M-FSK is a power-efficient, not bandwidth-efficient modulation scheme!**

Bandwidth Requirements of M-FSK

- Let $M = 2^{\lambda}$ and let the M signals be orthogonal. The minimum frequency separation between adjacent signals $\Delta f = \frac{R_s}{2}$.
- The bandwidth $B.W = (M-1)\left(\frac{R_s}{2}\right) + 2R_s$.
- For the case when $M = 2, B.W = \left(\frac{R_s}{2}\right) + 2R_s = \frac{5}{2}R_b$.
- For the case when $M = 4, B.W = \left(\frac{3R_s}{2}\right) + 2R_s = \frac{7}{2}R_s = \frac{7}{2}\frac{R_b}{\log(4)} = \frac{7}{4}R_b$.



Bit Error Probability of M-FSK

- Due to the symmetry of M-FSK constellation, all mappings from sequences of λ bits to signal points yield the same bit error probability.
- For equally likely signals, all the conditional error events are equiprobable and occur with probability $\Pr[\text{symbol error}]/(M-1) = \Pr[\text{symbol error}]/(2^\lambda - 1)$.
- There are $\binom{\lambda}{k}$ ways in which k bits out of λ may be in error $\Rightarrow$ The average number of bit errors per λ-bit symbol is

$$\sum_{k=1}^{\lambda} k \binom{\lambda}{k} \frac{\Pr[\text{symbol error}]}{2^\lambda - 1} = \lambda \frac{2^{\lambda-1}}{2^\lambda - 1} \Pr[\text{symbol error}].$$

- The probability of bit error is simply the above quantity divided by λ:

$$\Pr[\text{bit error}] = \frac{2^{\lambda-1}}{2^\lambda - 1} \Pr[\text{symbol error}].$$

M-ary Quadrature Amplitude Modulation (M-QAM)

- M-QAM are two-dim constellations and they involve inphase (I) and quadrature (Q) carriers:

$$\phi_I(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t), \quad 0 \le t \le T_s,$$
$$\phi_Q(t) = \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t), \quad 0 \le t \le T_s,$$

In M-QAM, the messages are encoded into both the amplitude and phase of the carrier.
QAM is a two-dimensional encoding scheme and requires two base functions.

- The ith transmitted M-QAM signal is:

$$\begin{aligned} s_i(t) &= V_{I,i}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t) + V_{Q,i}\sqrt{\frac{2}{T_s}}\sin(2\pi f_c t), \quad \begin{array}{l} 0 \le t \le T_s \\ i = 1, 2, \ldots, M \end{array} \\ &= \sqrt{E_i}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t - \theta_i) \end{aligned}$$

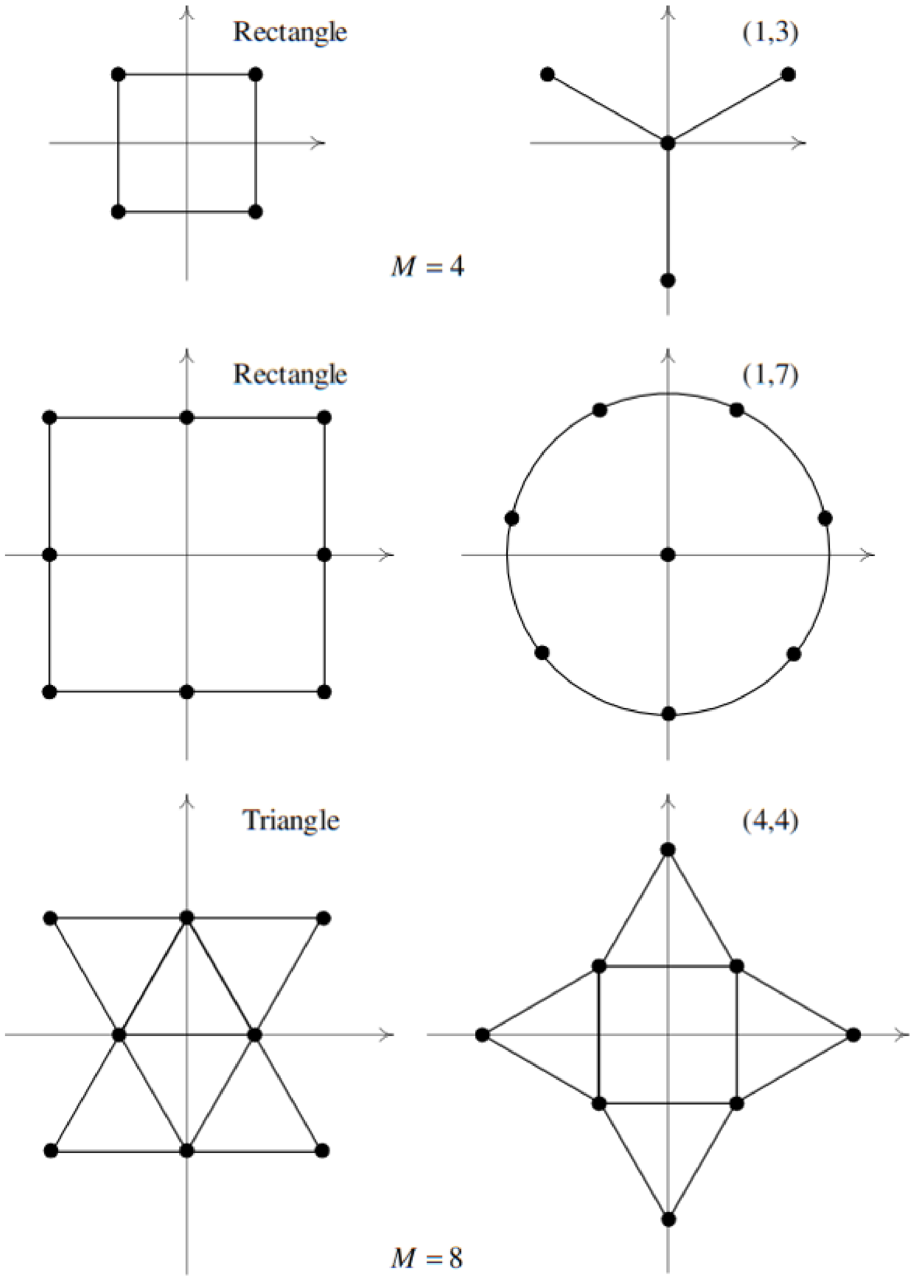
$$\boldsymbol{s_i(t) = a_i\phi_1 + b_i\phi_2}$$
$$\boldsymbol{E_i = {a_i}^2 + {b_i}^2}$$ **(prove)**

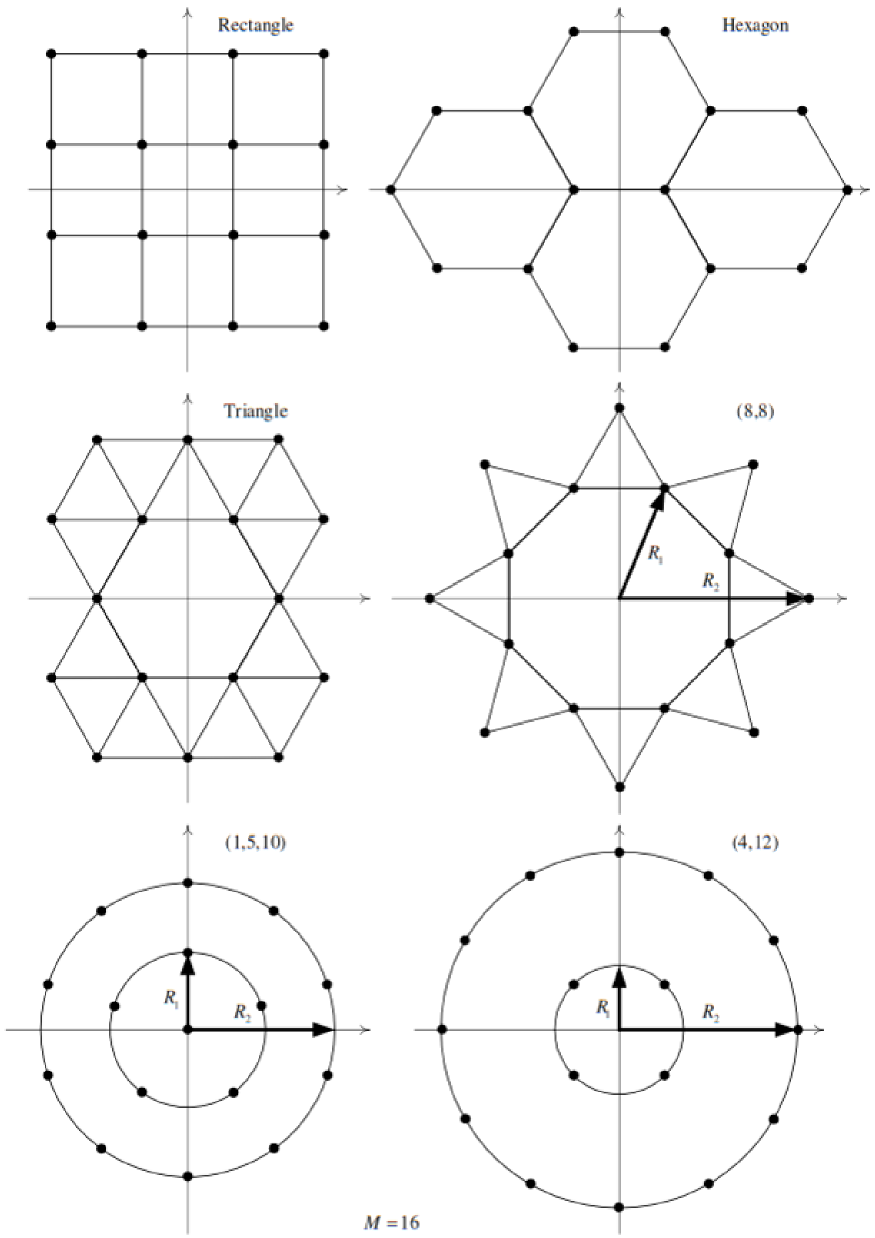
$V_{I,i}$ and $V_{Q,i}$ are the information-bearing discrete amplitudes of the two quadrature carriers, $E_i = V_{I,i}^2 + V_{Q,i}^2$ and $\theta_i = \tan^{-1}(V_{Q,i}/V_{I,i})$.

- In general, QAM symbols have different energies. The average symbol energy is calculated as:

$$E_s = \sum_{i=1}^{M} E_i P[s_i(t)] = \frac{\sum_{i=1}^{M} E_i}{M}, \quad \text{for equally-likely signals}$$

M-Ary Quadrature Amplitude Modulation



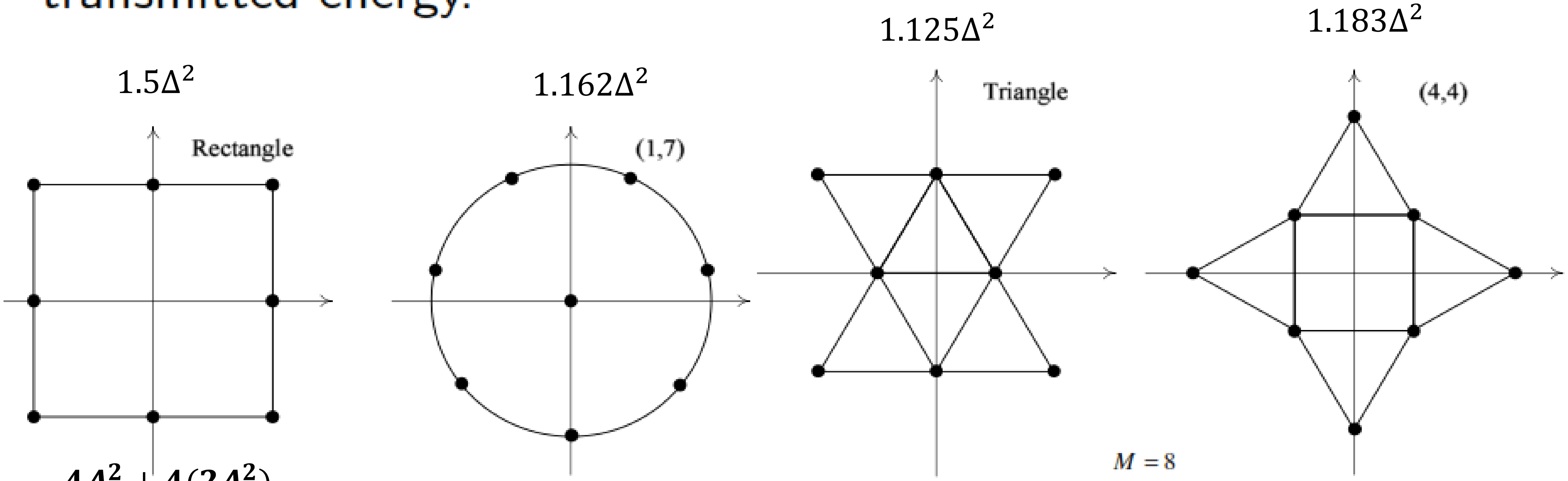


Criteria for Selecting a Given Constellation

- **Probability of Error**: In signaling over AWGN, the most likely errors are those which confuse a signal with its neighbors. To maintain the same symbol error probability, the distance between the nearest neighbors are kept the same.
- **Average Transmitted Energy**: The most efficient signal constellation is the one that has the smallest average transmitted energy.
- **Simplicity in Modulation and Demodulation.**
- **Bandwidth Requirement**.

A Simple Comparison of M-QAM Constellations

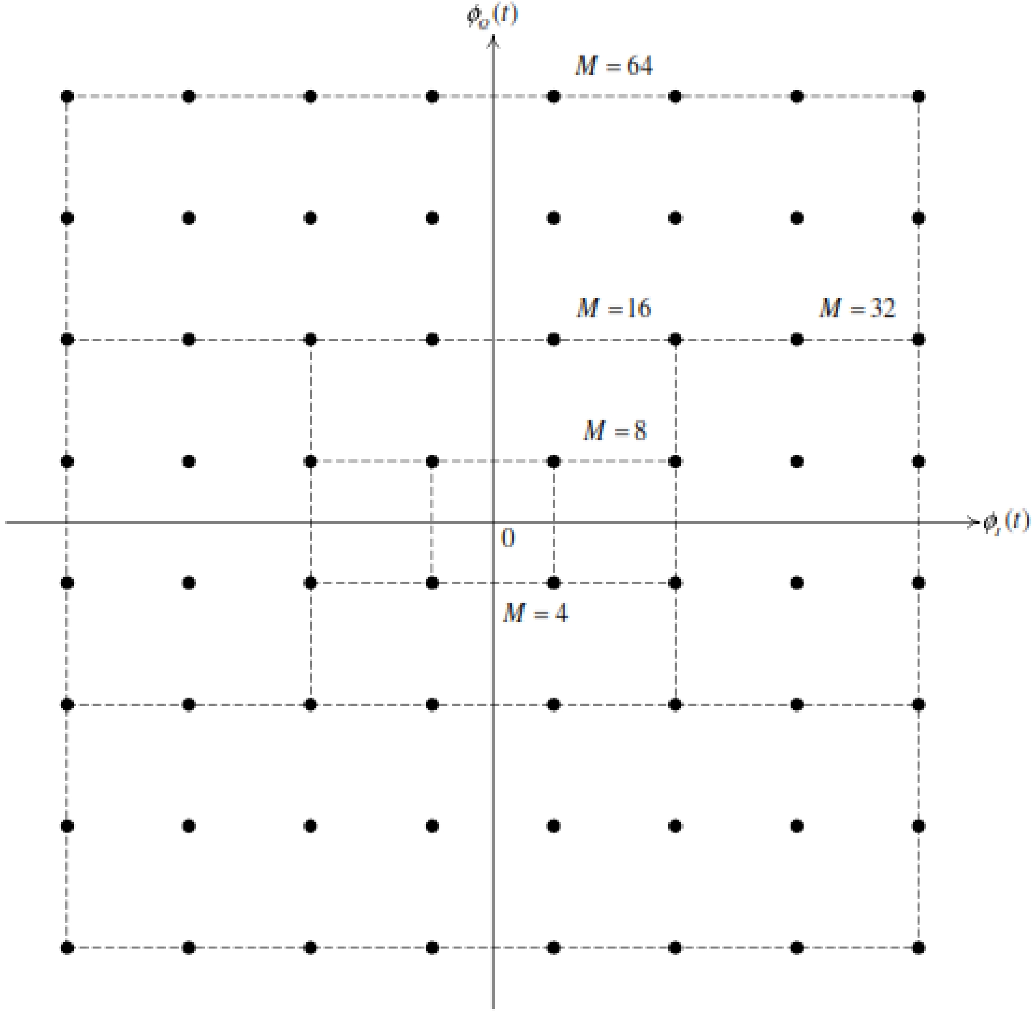
With the same *minimum* distance of all the constellations, a more efficient signal constellation is the one that has smaller average transmitted energy.



$$\boldsymbol{E_{av} = \frac{4\Delta^2 + 4(2\Delta^2)}{8} = 1.5\Delta^2}$$

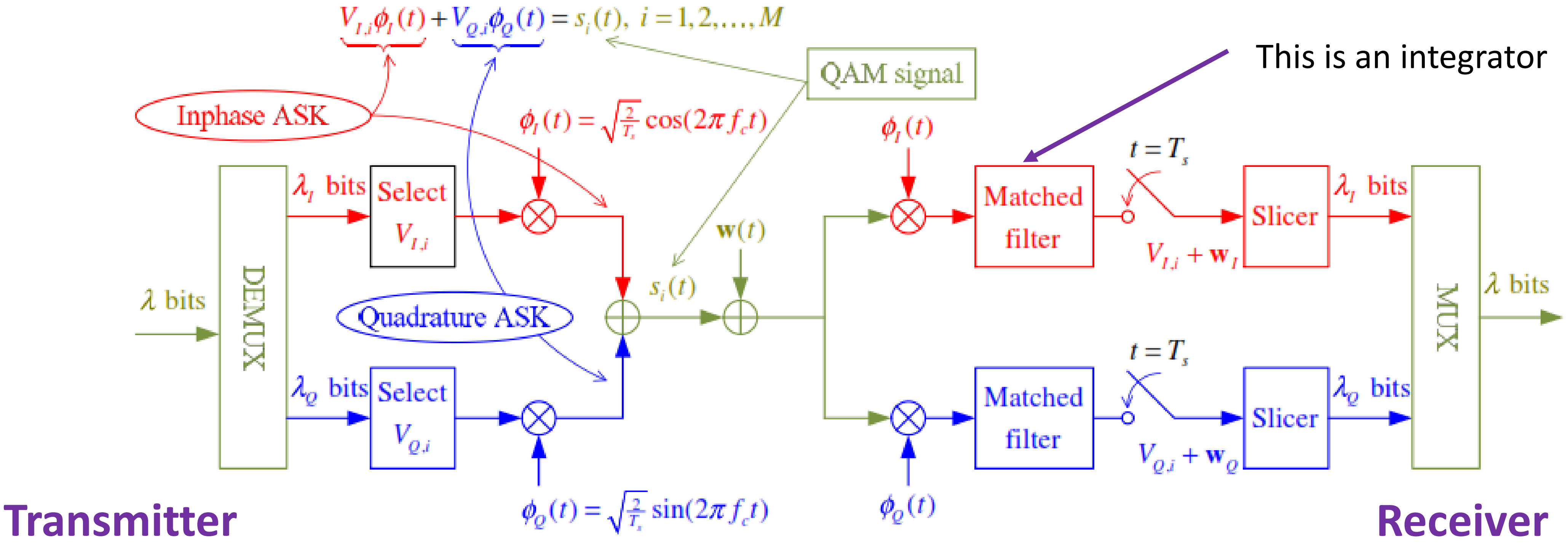
E_s for the rectangular, triangular, (1,7) and (4,4) constellations are found to be $1.50\Delta^2$, $1.125\Delta^2$, $1.162\Delta^2$ and $1.183\Delta^2$, respectively.

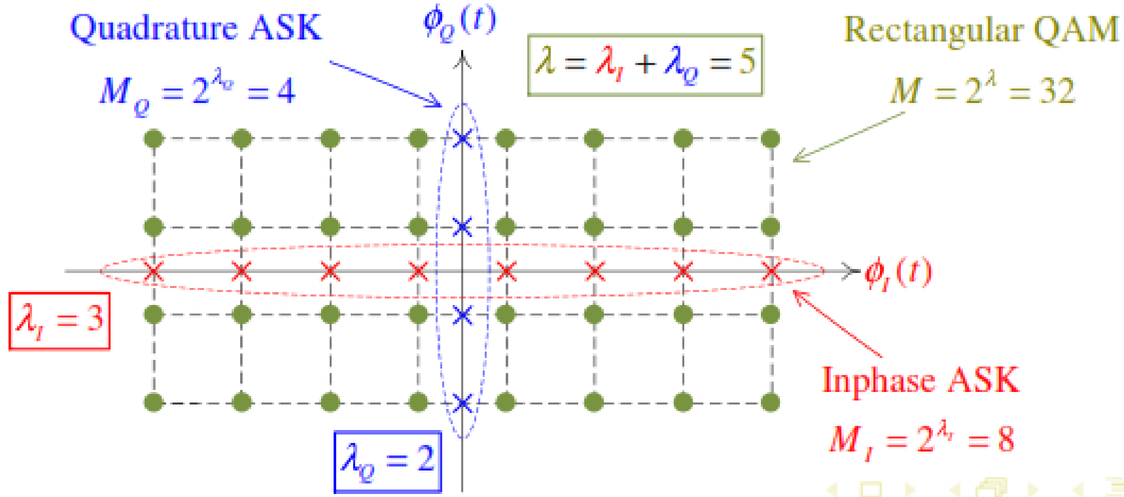
Rectangular M-QAM



- Signal *components* belong to the set of discrete values $\{(2i - 1 - M)\Delta/2\}$, $i = 1, 2, \ldots, \frac{M}{2}$.
- Each group of $\lambda = \log_2 M$ bits can be divided into λ_I inphase bits and λ_Q quadrature bits, where $\lambda_I + \lambda_Q = \lambda$.
- Inphase bits and quadrature bits modulate the inphase and quadrature carriers *independently*.

Implementation of Rectangular M-QAM

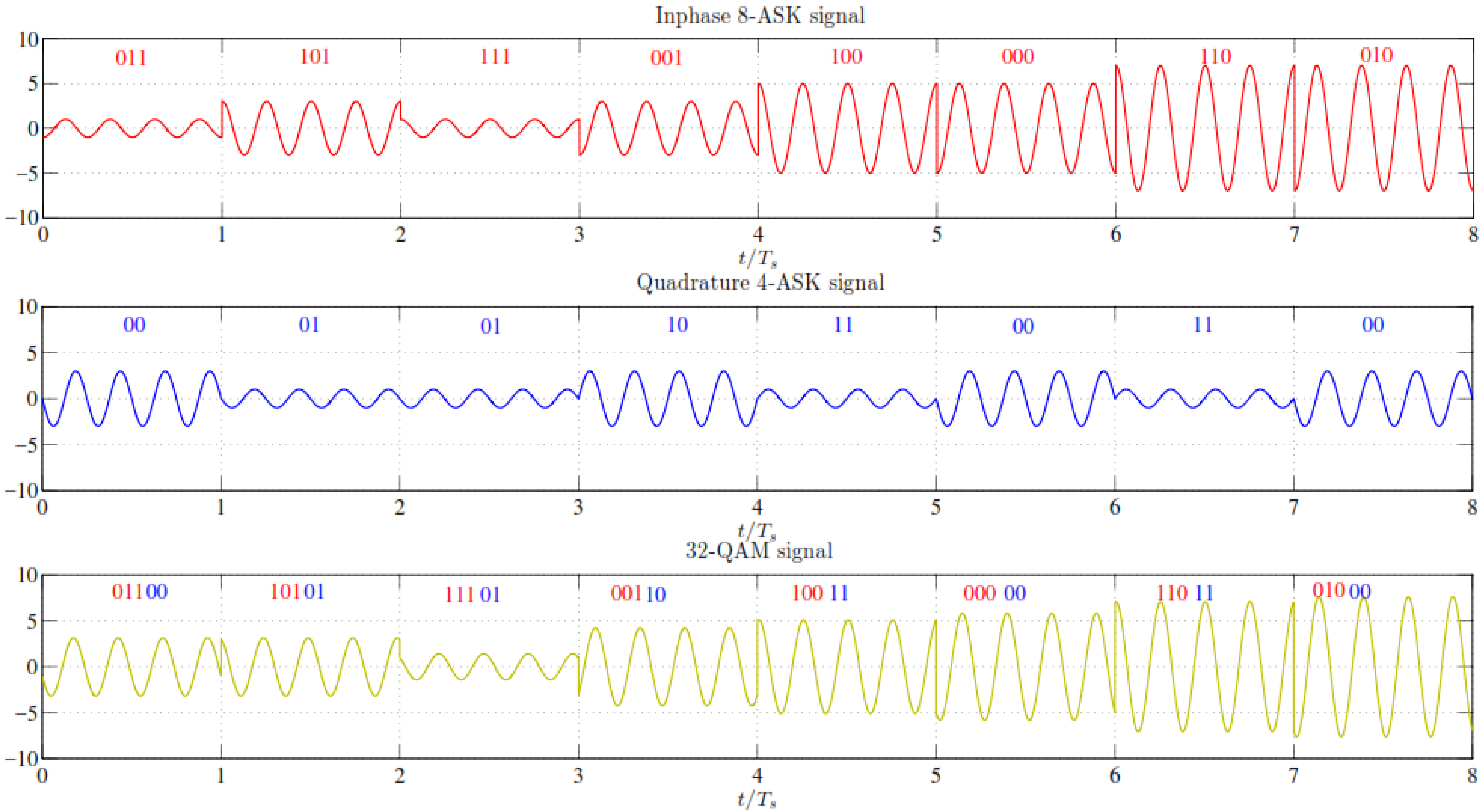




Each group of $\lambda = log_2 M$ can be divided into λ_I in-phase bits and λ_Q quadrature bits where $\lambda = \lambda_I + \lambda_Q$.

In-phase and quadrature bits modulate the in-phase and quadrature carriers independently.

In-phase ASK, Quadrature ASK and QAM Waveforms



Symbol Error Probability of M-QAM

- For square constellations:

$$P[\text{error}] = 1 - P[\text{correct}] = 1 - \left(1 - P_{\sqrt{M}}[\text{error}]\right)^2,$$

$$P_{\sqrt{M}}[\text{error}] = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right),$$

where E_s/N_0 is the average SNR per symbol.

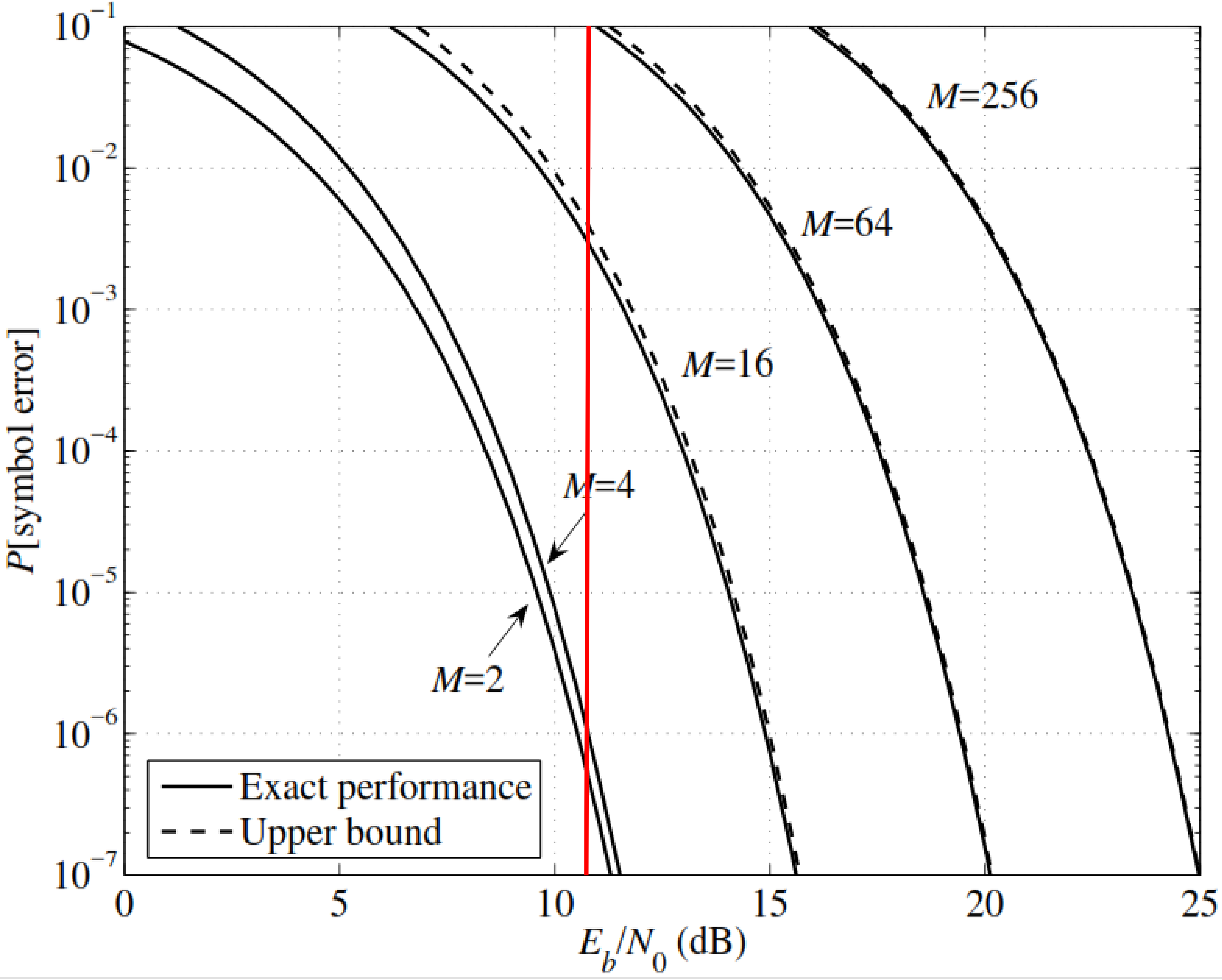
See the full derivation in Appendix 1.

- For general rectangular constellations:

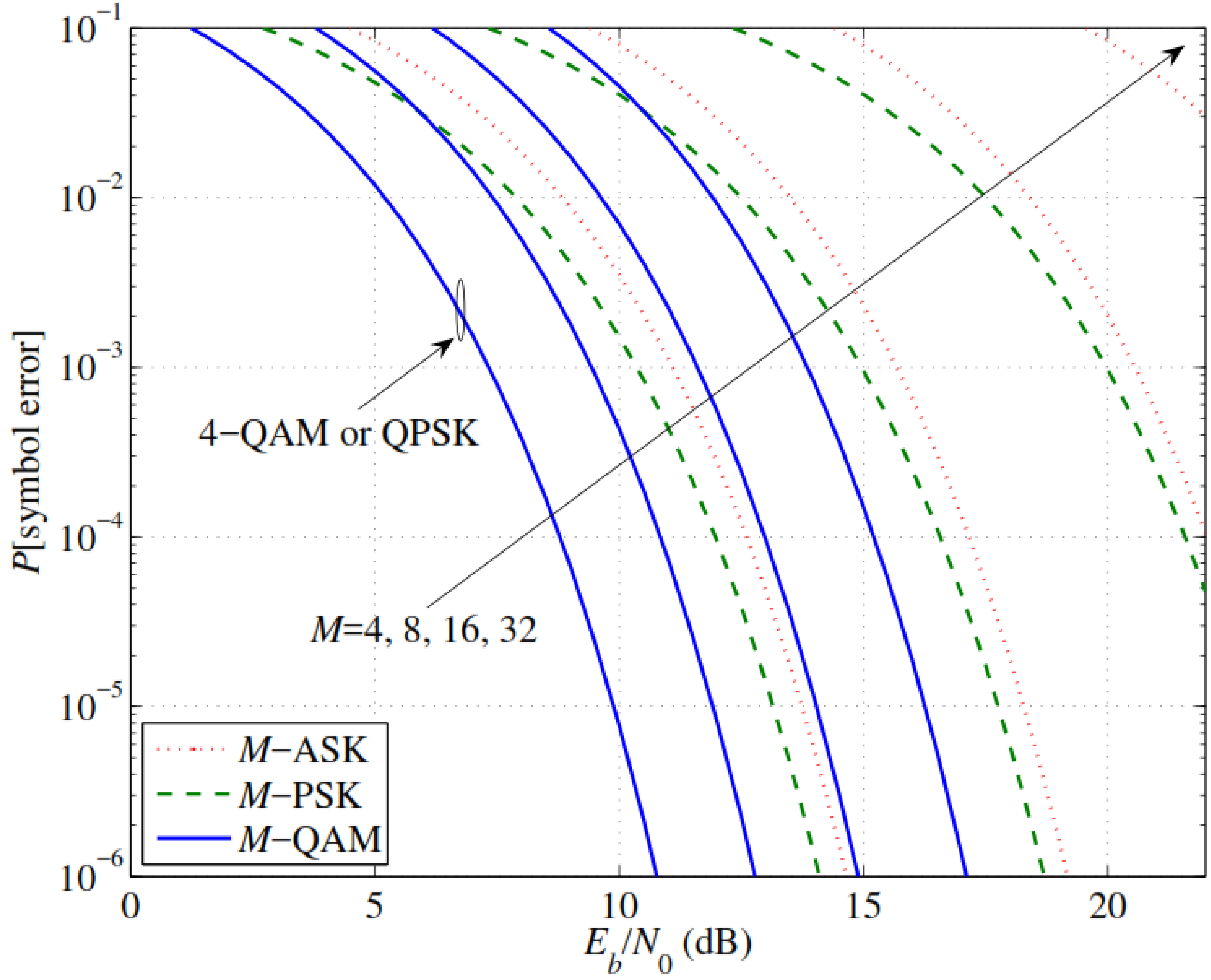
$$\begin{aligned} P[\text{error}] &\leq 1 - \left[1 - 2Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)\right]^2 \\ &\leq 4Q\left(\sqrt{\frac{3\lambda E_b}{(M-1)N_0}}\right) \end{aligned}$$

where E_b/N_0 is the average SNR per bit.

Symbol Error Probability of M-QAM



Performance Comparison of M-ASK, M-PSK, M-QAM



Comparison of M-ary Signaling Techniques

- A compact and meaningful comparison of different M-ary techniques is based on the bit rate–to bandwidth ratio, r_b/W (*bandwidth efficiency*) versus the SNR per bit, E_b/N_0 (*power efficiency*) required to achieve a given $P[\text{error}]$.
- To reduce bandwidth, M-ASK can be transmitted with a *single-sideband* (SSB) only. Thus the bandwidth can be approximated as $W \approx 1/(2T_s)$ and

$$\left(\frac{r_b}{W}\right)_{\text{SSB-ASK}} = 2\log_2 M \quad \text{(bits/s/Hz)}.$$

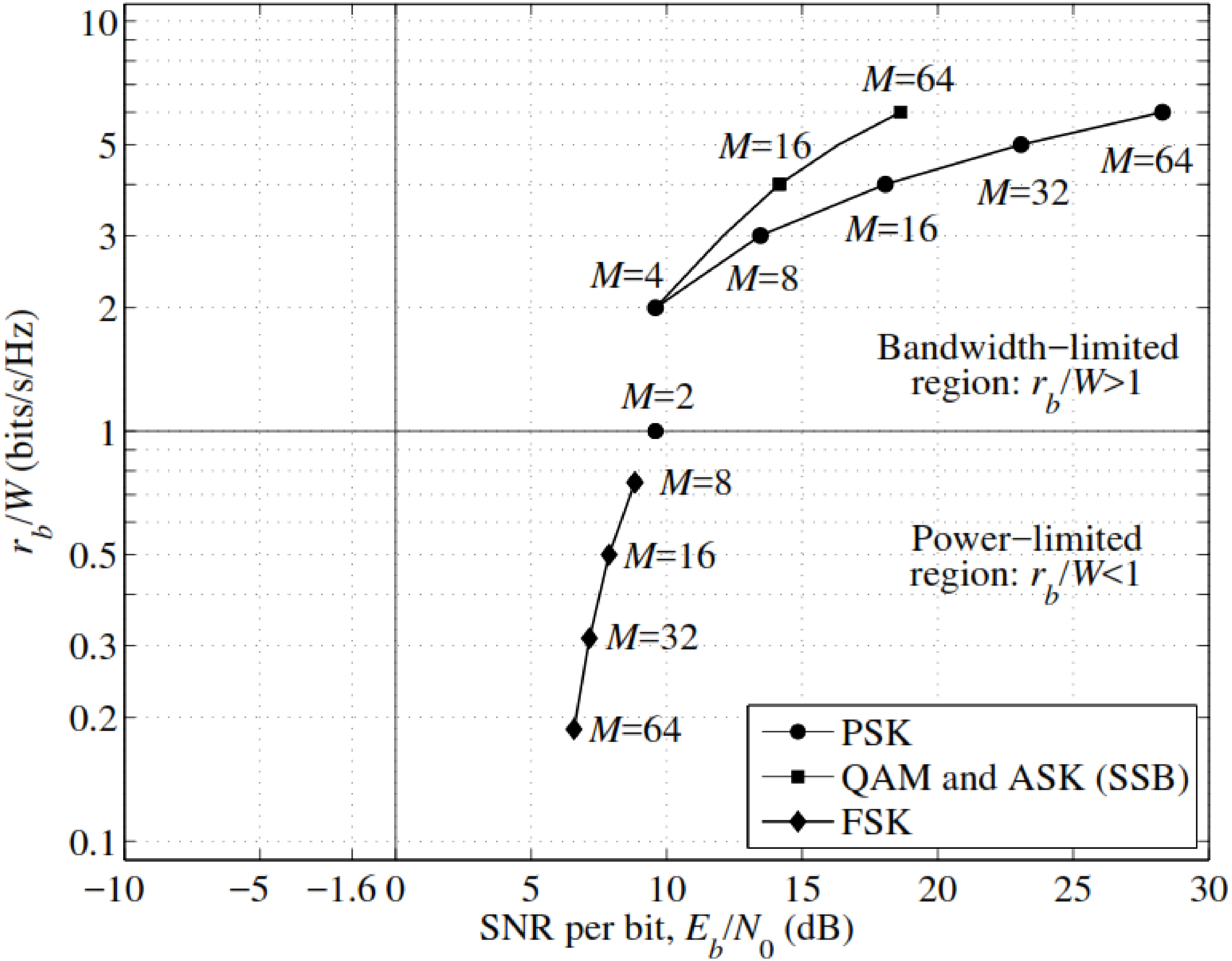
- M-PSK and M-QAM ($M > 2$) must be transmitted with *double sidebands*, hence $W \approx 1/T_s$ and

$$\left(\frac{r_b}{W}\right)_{\text{PSK}} = \log_2 M, \quad \text{(bits/s/Hz)},$$

- For M-FSK with the minimum frequency separation of $1/(2T_s)$, $W \approx \frac{M}{2T_s} = \frac{M}{2(\lambda/r_b)} = \frac{M}{2\log_2 M} r_b$, and

$$\left(\frac{r_b}{W}\right)_{\text{FSK}} = \frac{2\log_2 M}{M}.$$

Power-Bandwidth Plane (At P[error] =10^{-5})



Adaptive Modulation

- The probability of error (bit of symbol) has been obtained for various pass-band modulation schemes in terms of SNR per bit, E_b/N_0.
- Such probability analysis was obtained under the simplest channel model, namely AWGN. This channel model ignores any attenuation by the transmission medium/environment and only takes into account AWGN.
- In reality, there is always channel attenuation, even the attenuation is varying over time (think about wireless channels). Usually the received signal power is usually much smaller than the transmitted signal power.
- When considering channel attenuation, it is important to understand that it is the *received* SNR per bit, E_b/N_0, measured at the receiver side that determines the error probability.
- Consider bandwidth-efficient applications such as cellular phone or cable TV systems, in which the channel quality changes dynamically. There are two design options:
  1. In order to maintain the same transmission rate (i.e., stay with the same constellation size M) at the same quality of service (i.e., same error probability), the transmitted power should be adjusted according to the channel condition: **higher transmit power when the channel quality is poor and vice versa**.
  2. If the transmit power is held fixed, in order to maintain the same quality of service (i.e., same error probability), the transmitter needs to adapt the modulation scheme according to the channel condition: **lower constellation size (smaller M), which also means slower transmission rate, when the channel quality is poor and vice versa**.
- Many practical systems operate with constant transmit power (easier for circuit design) and exercise adaptive modulation.
- To enable adaptive modulation, the information about the channel quality, usually measured at the receiver, needs to be sent back to the transmitter on a reverse channel.
- For example, the latest DOCSIS 3.1 standard for cable TV specifies 14 modulation choices: BPSK, QPSK, 8-QAM, 16-QAM, 32-QAM, 64-QAM, 128-QAM, 256-QAM, 512-QAM, 1024-QAM, 2048-QAM, 4096-QAM, 8192-QAM, and 16384-QAM.

Union Bound

Union bound

The probability of a finite union of events is upper bounded by the sum of the probabilities of the individual events.

- Let A_{ki} denote that the observation vector Z is closer to the symbol vector s_k than s_i, when s_i is transmitted.
- $\mathrm{P}(A_{ki}) = P_2(s_k, s_i)$ depends only on s_k and s_i .
- Applying Union bounds yields

$$P_e(m_i) \le \sum_{\substack{k=1 \\ k \ne i}}^{M} P_2(s_k, s_i)$$

Assume all signals are equally likely. Therefore,
$P(E) = P_e(m_i)$
Recall that:
$P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$

Union Bound

$$P_2(s_k, s_i) = P(Z \text{ is closer to } s_k \text{ than } s_i \text{ when } s_i \text{ is sent}$$

$$= \int_{d_{ik}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{u^2}{N_0}\right) du = Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

$$d_{ik} = ||s_i - s_k||$$

$$P_e(m_i) \le \sum_{\substack{k=1 \\ k \ne i}}^{M} P_2(s_k, s_i) = \sum_{\substack{k=1 \\ k \ne i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

Example on the Union Bound: M-FSK

$$\boldsymbol{P_e(m_i) \le \sum_{\substack{k=1 \\ k\neq i}}^{M} P_2(s_k, s_i) = \sum_{\substack{k=1 \\ k\neq i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)}$$

Example: If all (M-1) signals are of the same distance d from s_i (as in M-FSK), the error probability of error is approximately

$$P_e(m_i) \le \sum_{\substack{k=1 \\ k\neq i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right) \approx (M-1)\, Q\left(\frac{d}{\sqrt{2N_0}}\right)$$

where, $d = \sqrt{2E_s}$. Therefore,

$$P_e(m_i) \approx (M-1)Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right) = (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

Example on the Union Bound: 4-PSK

$\|\mathbf{s}_i\| = \sqrt{E_i} = \sqrt{E_s},\ i = 1, \ldots$

$$P_e(m_i) \approx \sum_{\substack{k=1 \\ k \neq i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

$$P_e(m_i) \approx Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right) + Q\left(\frac{d_{14}}{\sqrt{2N_0}}\right) + Q\left(\frac{d_{14}}{\sqrt{N_0}}\right)$$

$$P_e(m_i) \approx 2Q\left(\frac{\sqrt{2E}}{\sqrt{2N_0}}\right) + Q\left(\frac{2\sqrt{E}}{\sqrt{2N_0}}\right)$$

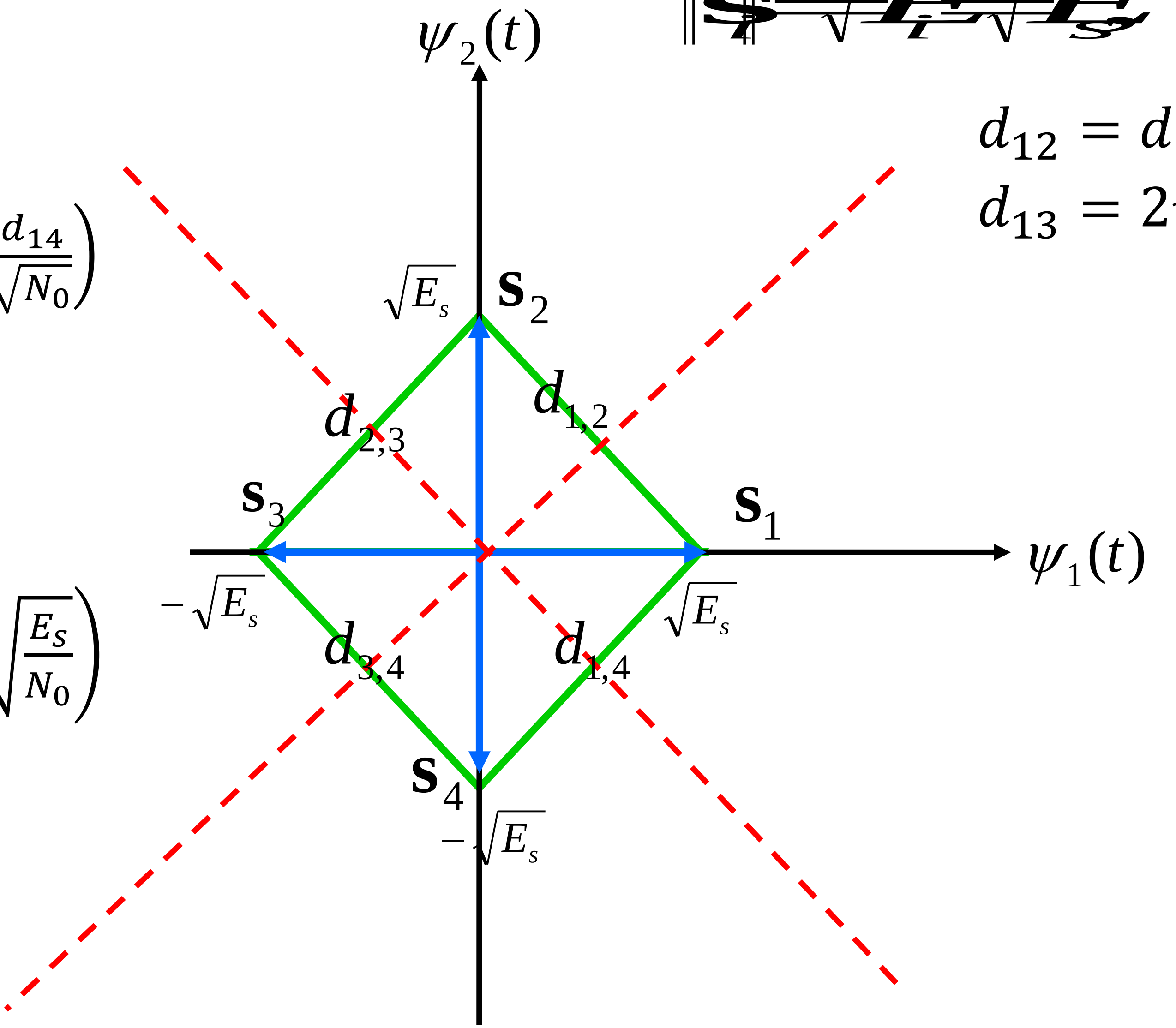
$$P_e(m_i) \approx 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) + Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

Note that since $Q\left(\sqrt{\frac{2E_s}{N_0}}\right) \ll 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

$$P_e(m_i) \approx 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$d_{12} = d_{14} = \sqrt{2E}$

$d_{13} = 2\sqrt{E}$



Example on the Union Bound: M-QAM

- In M-QAM, each signal has 4 signals at a distance Δ and 4 at a distance $\Delta\sqrt{2}$.
- The probability of error can be approximated by:

$P_e(m_i) \approx 4Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)+4Q\left(\frac{\Delta\sqrt{2}}{\sqrt{2N_0}}\right)$; By ignoring the second term, we get

$$P_e(m_i) \approx 4Q\left(\sqrt{\frac{\Delta^2}{2N_0}}\right);$$

- The average energy per signal is (See Appendix 1)

$$E_s = \frac{2(M-1)\Delta^2}{12}; \text{ (2)}$$

- Substituting for Δ (in 2), into (1), we get

$$P_e(m_i) \approx 4Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right); \text{ as was found earlier}$$